

# ENGINEERING ECONOMY

subscribe on youtube & fb  
@engineerdmath

SIMPLE INTEREST - only the principal earns interest

$$I = Pni$$

$$F = P + I$$

$$F = P + Pni$$

$$F = P(1 + ni)$$

where:  $I$  = interest

$i$  = rate of interest per interest period

$n$  = number of interest period

$P$  = principal or present worth

$F$  = accumulated amount or future worth

Note:

- ordinary simple interest is computed on the basis of 12 months of 30 days each year or 360 days a year.
- exact simple interest is based on the exact number of days in a year, 365 days for an ordinary year and 366 days for a leap year.

31-day months

January

March

May

July

August

October

December

30-day months

April

June

September

November

February - 28/29 days

\* To find whether a year is ordinary year or leap year, if it is divisible by 4, then it is a leap year, otherwise ordinary year.

### Sample Problem 1:

What is the exact simple interest of ₱1500 for the period from January 10 to October 28, 1980 at 15% rate of interest.

Solution:

Given:

$n$  = from January 10 to October 28, 1980

$i = 15\%$

$P = ₱1500$

Check if 1980 is a leap year/ ordinary year

$$\frac{1980}{4} = 495 \quad \therefore \text{Leap year}$$

January 10 → 21

February → 29

March → 31

April → 30

May → 31

June → 30

July → 31

August → 31

September → 30

October 28 → + 28  
292 days

$$F = P(1 + in) \\ = ₱1500 \left( 1 + 0.15 \cdot \frac{292}{366} \right)$$

$$F = ₱1679.5$$

## Cash Flow Diagram

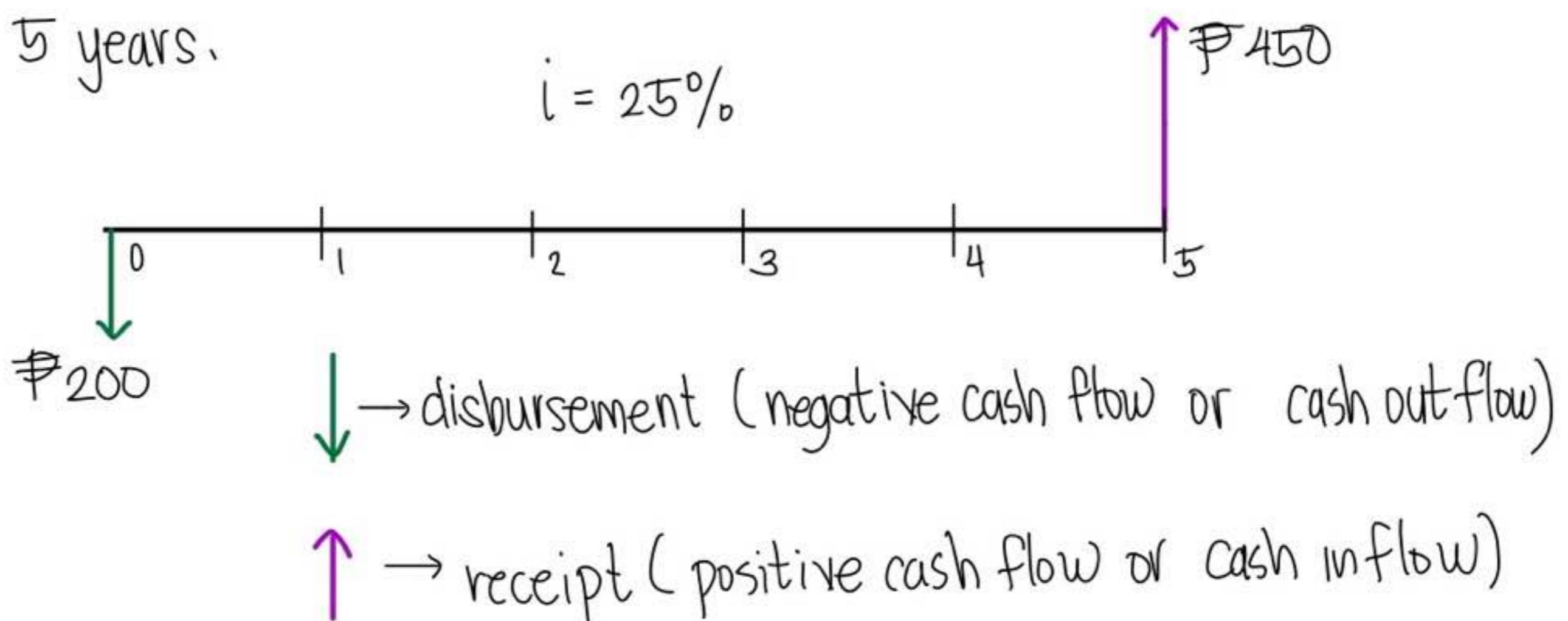
subscribe on youtube & fb  
@enginerdmath

- a graphical representation of cash flows drawn on a time scale.

ex: A loan of ₱200 at a simple interest of 25% will become ₱450

after 5 years.

$$i = 25\%$$



## COMPOUND INTEREST

- both principal and interest earn interest
- "interest on top of interest"

$$F = P(1+i)^{n-1} + P(1+i)^{n-i} i$$

$\downarrow$   
 principal at the  
 beginning of period n

$\downarrow$   
 interest earned  
 during period n

$$= P(1+i)^{n-1} (1+i) \rightarrow \text{factoring out } (1+i)^{n-1}$$

$$\therefore F = P(1+i)^n \rightarrow \text{amount at the end of period } n.$$

$(1+i)^n \rightarrow$  "single payment compound amount factor"  $(F/P, i\%, n)$

$$\therefore F = P(F/P, i\%, n)$$

$\downarrow$   
 "F given P at  $i\%$  in  
 n interest period"

Also:

$$P = F(1+i)^{-n}$$

→ principal/present worth amount of money

$$(1+i)^{-n} \rightarrow \text{"single payment present worth factor"} (P/F, i\%, n)$$

$$P = F(P/F, i\%, n)$$

↓  
 "P given F at  $i\%$  in  
 n interest period."

## RATES OF INTEREST

### a) Nominal rate of interest

- specifies rate of interest and a number of interest periods in one year

where:

$$i = \frac{r}{m}$$

$i$  = rate of interest per interest period

$r$  = nominal rate of interest

$m$  = number of compounding periods  
per year:

As a consequence:

$$\therefore F = P(i + \frac{r}{m})^{mn}$$

$$P = F(1 + \frac{r}{m})^{-mn}$$

note: annual  $\rightarrow m=1 \therefore r=i$

semiannual  $\rightarrow m=2$

quarterly  $\rightarrow m=4$

bimonthly  $\rightarrow m=6$

monthly  $\rightarrow m=12$

### b) Effective rate of interest

- actual interest rate of an investment in one year or the interest rate for any compounding that will give the same accumulation as computed compounded annually.

$$E.I. = (1+i)^m - 1 \text{ or}$$

$$E.I. = \left(1 + \frac{r}{m}\right)^m - 1$$

subscribe on youtube & fb  
@enginerdmath

### Sample Problem 2:

- 1) What is the equivalent nominal interest rate compounded quarterly of 12% compounded semiannually?

Solution:

Given:

Compounding quarterly

Compounding semiannually

$$r = ?$$

$$r = 12\%$$

$$m = 4$$

$$m = 2$$

Equating effective rate of interest:



$$E.I. = \left(1 + \frac{r}{m}\right)^m - 1 \rightarrow \left(1 + \frac{r}{4}\right)^4 - 1 = \left(1 + \frac{0.12}{2}\right)^2 - 1$$

Solving for r:

$$r = 0.11825 \text{ or } 11.825\%$$

### Sample Problem 3:

- 1) Find the amount at the end of two years and seven months if ₱1000 is invested at 8% compounded quarterly. Using simple interest for any time less than a year with interest period at 10%.

Solution:

Given: For compound interest

$$P = \text{P}1000$$

$$n = 2 \text{ years}$$

$$r = 8\%$$

$$m = 4 \text{ (compounded quarterly)}$$

For simple interest

$$n = 7 \text{ months (less than a year)}$$

$$= \frac{7}{12} \text{ years}$$

$$i = 10\%$$

For the first two years:

$$F_1 = P \left(1 + \frac{r}{m}\right)^{mn} = \text{P}1000 \left(1 + \frac{0.08}{4}\right)^{(4)(2)} = \text{P}1171.66$$

$$\begin{aligned} F_2 &= P(1 + ni) \\ &\downarrow F_1 \\ &= \boxed{\text{P}1240} \end{aligned}$$

## EQUATION OF VALUES

- it is the equation formed in setting one set of obligation equal to another set of obligation brought to any point in a cash flow diagram known as the focal date or comparison date or reference point.

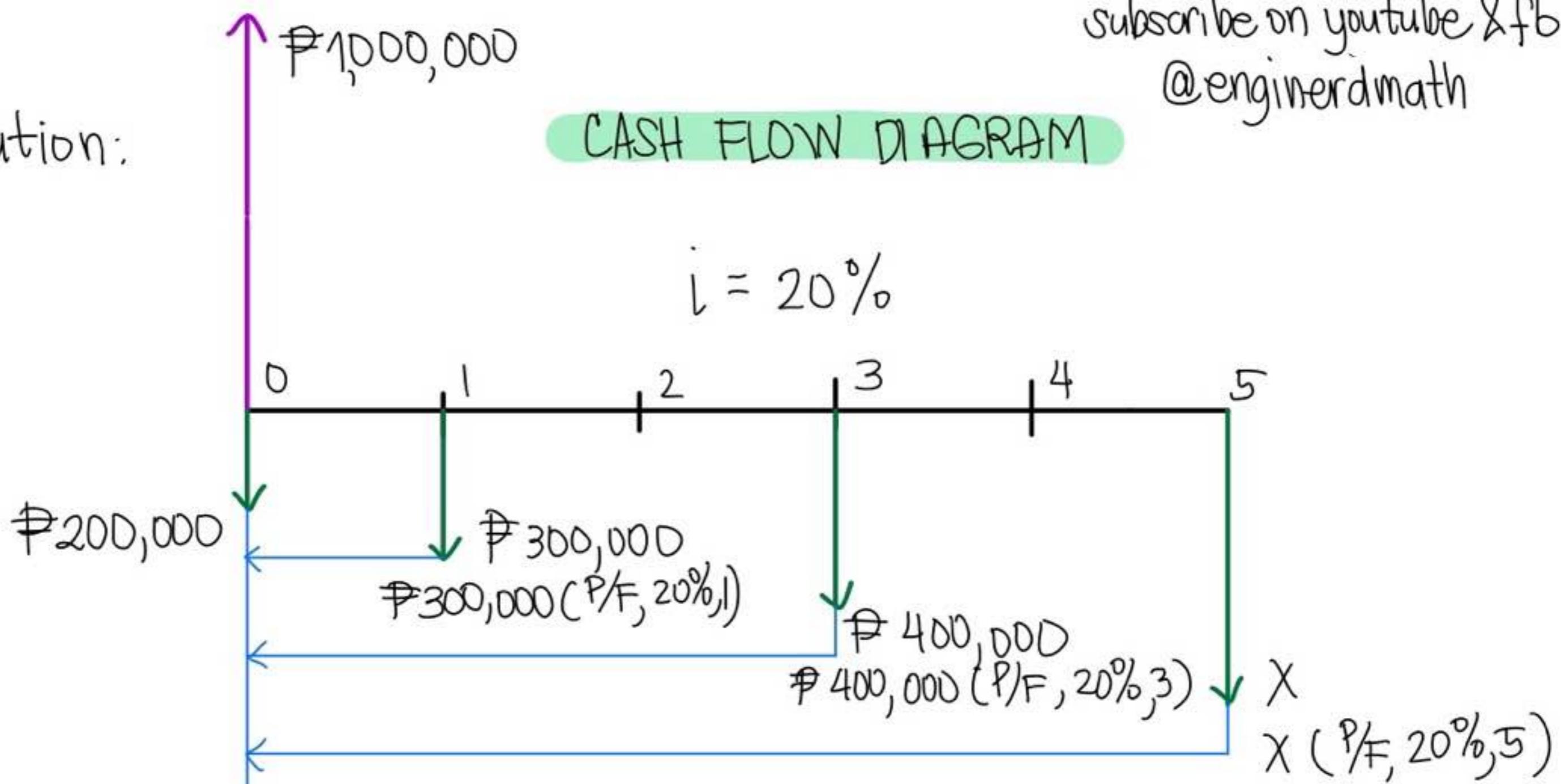
### Sample Problem 4:

- i) A man bought a lot worth ₱1,000,000 if paid in cash. On the installment basis, he paid a downpayment of ₱200,000; ₱300,000 at the end of one year, ₱400,000 at the end of three years and a final payment at the end of five years. What was the final payment if interest was 20%.

7

subscribe on youtube & fb  
@enginerdmath

Solution:



Setting focal date to 0;

$$\sum \text{cash inflow} = \sum \text{cash outflow}$$

$$\text{₱}1,000,000 = \text{₱}200,000 + \text{₱}300,000(1+0.2)^{-1} + \text{₱}400,000(1+0.2)^{-3} + X(1+0.2)^{-5}$$

Solving for  $X$ :

$$X = \text{₱} 792.576$$

## DISCOUNT

- in a negotiable paper, it is the difference between the future worth and the present worth.
- interest paid in advance

$$D = F - P$$

where:  $D$  = discount

$d$  = discount rate per period

$$d = \frac{D}{F}$$

$$d = \frac{i}{1+i}$$

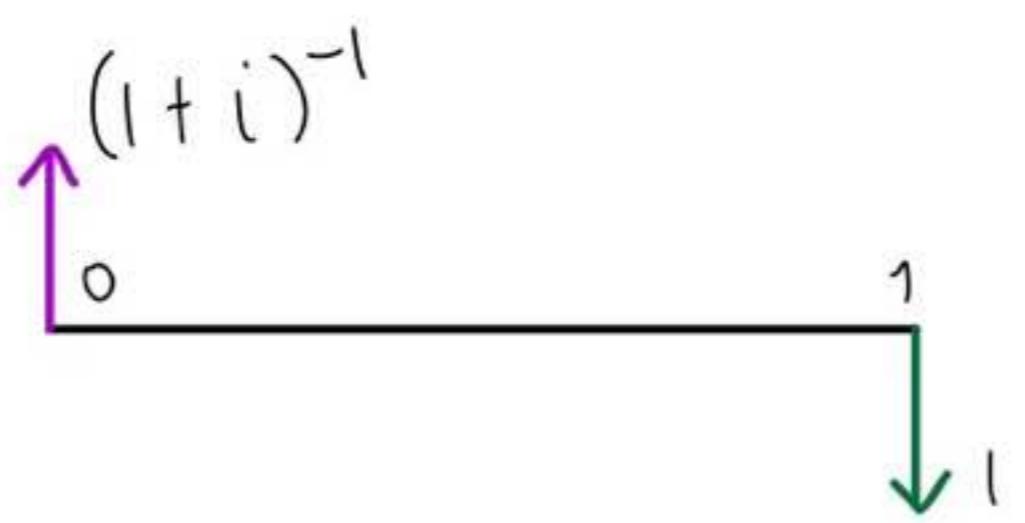
$$i = \frac{d}{1-d}$$

$$i = \frac{D}{P}$$

$$d = \frac{F-P}{F}$$

## Derivation:

The rate of discount is the discount on one unit of principal for one unit of time.



$$d = 1 - (1+i)^{-1}$$

$$d = \frac{i}{1+i}$$

$$i = \frac{d}{1-d}$$

## Sample Problem 5:

Tata made a loan from his employer an amount of ₦100,000 with a rate of simple interest of 20%, but the interest was deducted from the loan when the money was borrowed. If at the end of one year, Tata has to pay the full amount of ₦100,000, determine the actual rate of interest.

## Solution:

Given:  $d = 20\%$ ,  $F = ₦100,000$ ,  $i = ?$

## Solution 1:

$$i = \frac{d}{1-d} = \frac{0.20}{1-0.20} = 0.25 \text{ or } 25\%$$

## Solution 2:

$$D = Fd = ₦100,000 (0.20) = ₦20,000$$

$$P = F - D = ₦100,000 - ₦20,000 = ₦80,000$$

$$i = \frac{D}{P} = \frac{₦20,000}{₦80,000} = 0.25 \text{ or } 25\%$$

# ANNUITIES

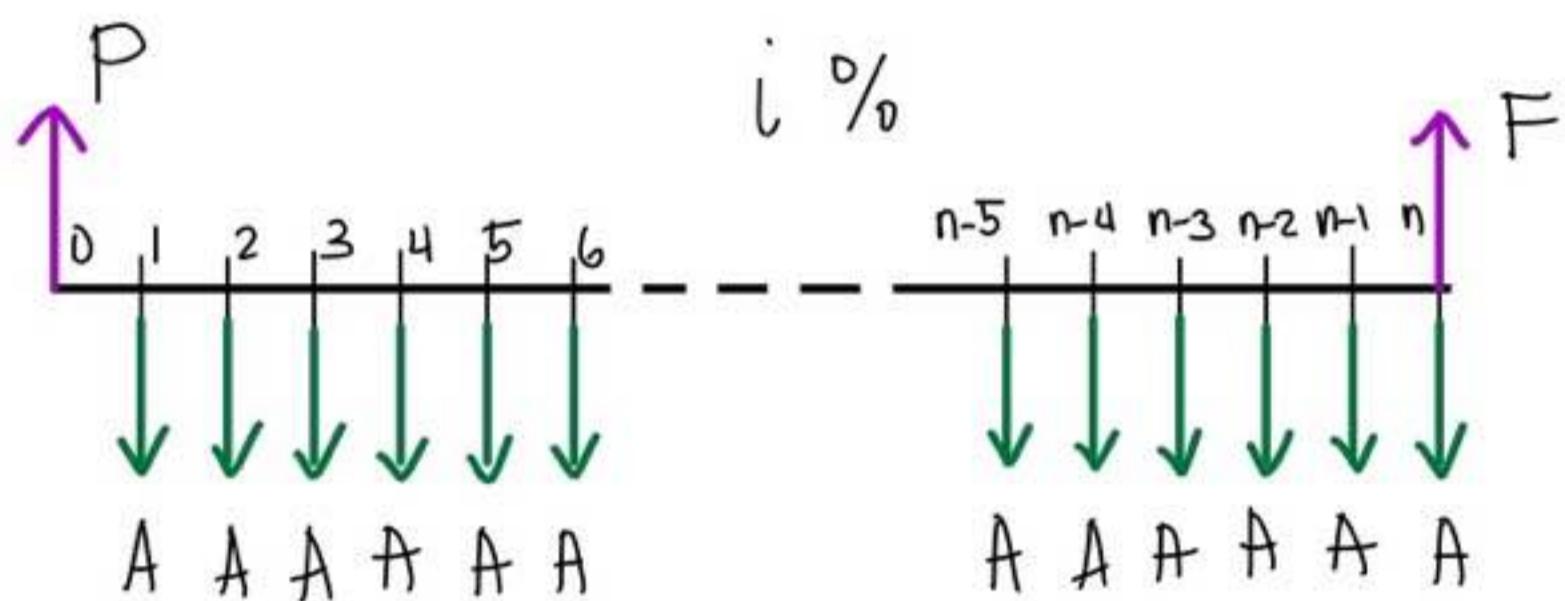
- a series of equal payments occurring at equal periods of time

## Types of Annuity

### D) Ordinary Annuity

- payments are made at the end of each period.

#### CASH FLOW DIAGRAM



To Find  $F$ , use  $n$  as the focal date:

$$F = A + A(1+i)^1 + A(1+i)^2 + A(1+i)^3 + \dots + A(1+i)^{n-2} + A(1+i)^{n-1}$$

$$= A \underbrace{\left[ 1 + (1+i)^1 + (1+i)^2 + (1+i)^3 + \dots + (1+i)^{n-2} + (1+i)^{n-1} \right]}_{\text{This is a geometric series}}$$

This is a geometric series:

$$S_G = a_1 \left[ \frac{r^n - 1}{r - 1} \right]$$

$$\therefore a_1 = 1, r = (1+i), n = n \quad n = \text{number of terms}$$

$S_G$  = sum of geometric series

$a_1$  = first term

$r$  = common ratio

$n$  = number of terms

$$S_G = (1) \left[ \frac{(1+i)^n - 1}{1+i - 1} \right]$$

$$F = A \cdot S_G = \frac{(1+i)^n - 1}{i}$$

As consequence,  $i = \frac{r}{m}$ ,

$$F = A \left[ \frac{(1+i)^n - 1}{i} \right] \quad \text{or}$$

$$F = A \left[ \frac{\left(1 + \frac{r}{m}\right)^{mn} - 1}{\frac{r}{m}} \right]$$

To find P, use 0 as focal date:

$$P = A(1+i)^{-1} + A(1+i)^{-2} + A(1+i)^{-3} + \dots + A(1+i)^{-(n-1)} + A(1+i)^{-n}$$

$$= A(1+i)^{-1} \left[ 1 + (1+i)^{-1} + (1+i)^{-2} + \dots + (1+i)^{-(n-2)} + (1+i)^{-(n-1)} \right]$$

This is a geometric series

Using  $S_G$  formula above again, where:

$$a_1 = 1, r = (1+i)^{-1}, n = n$$

$$S_G = (1) \left[ \frac{(1+i)^{-n} - 1}{(1+i)^{-1} - 1} \right] = \frac{(1+i)^{-n} - 1}{(1+i)^{-1} - 1}$$

$$\therefore P = A(1+i)^{-1} S_G$$

$$= A(1+i)^{-1} \left[ \frac{(1+i)^{-n} - 1}{(1+i)^{-1} - 1} \right]$$

Multiply both by  $\frac{(1+i)}{(1+i)}$

$$P = A(1+i)^{-1} \left[ \frac{(1+i)^{-n} - 1}{(1+i)^{-1} - 1} \right] \cdot \frac{(1+i)}{(1+i)}$$

$$= A \frac{(1+i)^{-n} - 1}{1 - (1+i)}$$

$$= A \frac{(1+i)^{-n} - 1}{-\dot{i}}$$

$$P = A \left[ \frac{1 - (1+i)^{-n}}{\dot{i}} \right]$$

or

$$\text{As consequence, } \dot{i} = \frac{r}{m}$$

$$P = A \left[ \frac{1 - (1 + \frac{r}{m})^{-mn}}{\frac{r}{m}} \right]$$

In  $F = A \left[ \frac{(1+i)^n - 1}{i} \right]$ , the factor  $\frac{(1+i)^n - 1}{i}$  is called "uniform series compound factor" ( $F/A, i\%, n$ )  $\rightarrow F$  given  $A$  at  $i\%$  in  $n$  interest periods.

$$\therefore F = A (F/A, i\%, n)$$

In  $P = A \left[ \frac{1 - (1+i)^{-n}}{i} \right]$ , the factor  $\frac{1 - (1+i)^{-n}}{i}$  is called the "uniform series present worth factor" ( $P/A, i\%, n$ ) read as  $P$  given  $A$  at  $i\%$  in  $n$  interest periods.

$$\therefore P = A (P/A, i\%, n)$$

Also, in  $A = F \left[ \frac{i}{(1+i)^n - 1} \right]$ , the factor  $\frac{i}{(1+i)^n - 1}$  is called the "sinking fund factor" ( $A/F, i\%, n$ ) read as  $A$  given  $F$  at  $i\%$  in  $n$  interest periods.

$$\therefore A = F (A/F, i\%, n)$$

and in  $A = P \left[ \frac{i}{1 - (1+i)^{-n}} \right]$ , the factor  $\frac{i}{1 - (1+i)^{-n}}$  is called the

"capital recovery factor" ( $A/P, i\%, n$ ) read as  $A$  given  $P$  at  $i\%$  in  $n$  interest periods.  $\therefore A = P (A/P, i\%, n)$

Relationship between capital recovery factor and sinking fund factor:

$$\frac{i}{(1+i)^n - 1} + i = \frac{i + i(1+i)^n - i}{(1+i)^n - 1}$$

$$= \frac{i(1+i)^n}{(1+i)^n - 1} \cdot \frac{(1+i)^n}{(1+i)^n}$$

subscribe on youtube & fb  
@engineerdmath

$$\frac{i}{(1+i)^n - 1} + i = \frac{i}{1 - (1+i)^{-n}}$$

$\therefore$  sinking fund factor + interest rate = capital recovery factor

$$(A/F, i\%, n) + i = (A/P, i\%, n)$$

### Sample Problem 6

The purchase amount of an equipment ₱ 100,000 has been made available through a loan which earns 12% per annum. It has been agreed that the loan be payable in 10 equal payments. How much then is the yearly due?

Solution:

Given:  $P = ₱ 100,000$ ,  $i = 12\%$ ,  $n = 10$  years,  $A = ?$

$$P = A \left[ \frac{1 - (1+i)^{-n}}{i} \right]$$

$$₱ 100,000 = A \left[ \frac{1 - (1+0.12)^{-10}}{0.12} \right]$$

Solving for A:

$$A = ₱ 17,698.42$$

## Sample Problem 7

What is the future worth of ₱600 deposited at the end of every month for 4 years if the interest rate is 12% compounded quarterly?

Solution:

$$\text{Given: } A = \text{₱}600$$

$$\begin{array}{l} \text{compounded quarterly} \\ \left\{ \begin{array}{l} r = 12\% \\ m = 4 \\ n = 4 \text{ years} \\ F = ? \end{array} \right. \end{array} \rightarrow \text{need to convert to monthly since the payment is every month.}$$

$$\begin{array}{l} m = 12 \\ r = ? \end{array}$$

Convert 12% compounded quarterly to  $r\%$  compounded monthly:

$$\text{e.i. monthly} = \text{e.i. quarterly}$$

$$\left(1 + \frac{r}{12}\right)^{12} - 1 = \left(1 + \frac{0.12}{4}\right)^4 - 1$$

Solving for  $r$ :  $r = 0.1188$  or 11.88% compounded monthly

Now get  $F$ :

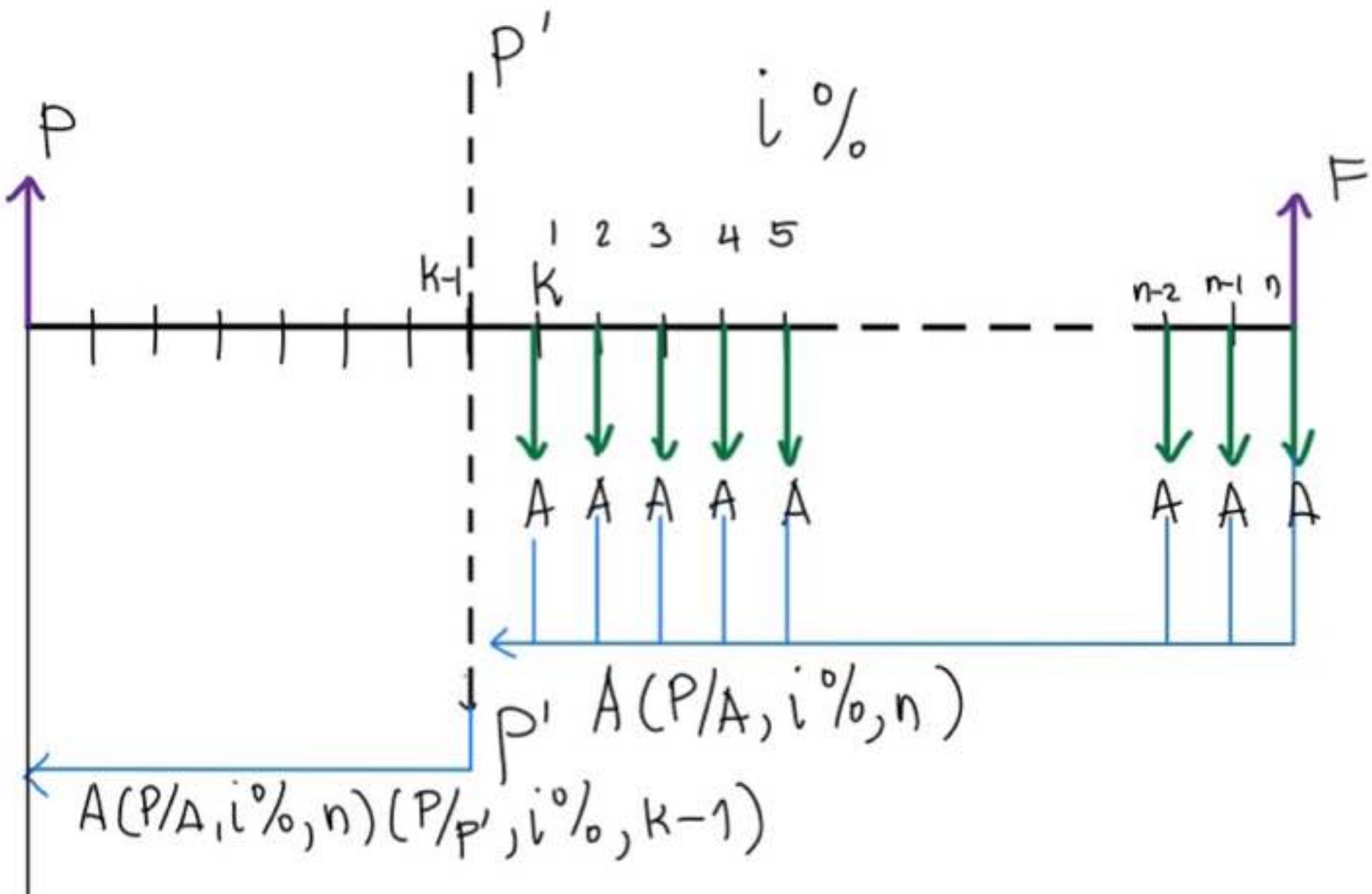
$$\begin{aligned} F &= A \left[ \frac{\left(1 + \frac{r}{m}\right)^{mn} - 1}{\frac{r}{m}} \right] \\ &= \text{₱}600 \left[ \frac{\left(1 + \frac{0.1188}{12}\right)^{12(4)} - 1}{\frac{0.1188}{12}} \right] \end{aligned}$$

$$F = \text{₱} 36,641.32$$

## 2) Deferred Annuity

- is the type of annuity where the payments are deferred for several periods of time

## CASH FLOW DIAGRAM



$$P' = A \left[ \frac{1 - (1+i)^{-n}}{i} \right]$$

$$P = P'(1+i)^{-(k-1)}$$

where  $k$  = deferment period

$$P = A \left[ \frac{1 - (1+i)^{-n}}{i} \right] (1+i)^{-(k-1)}$$

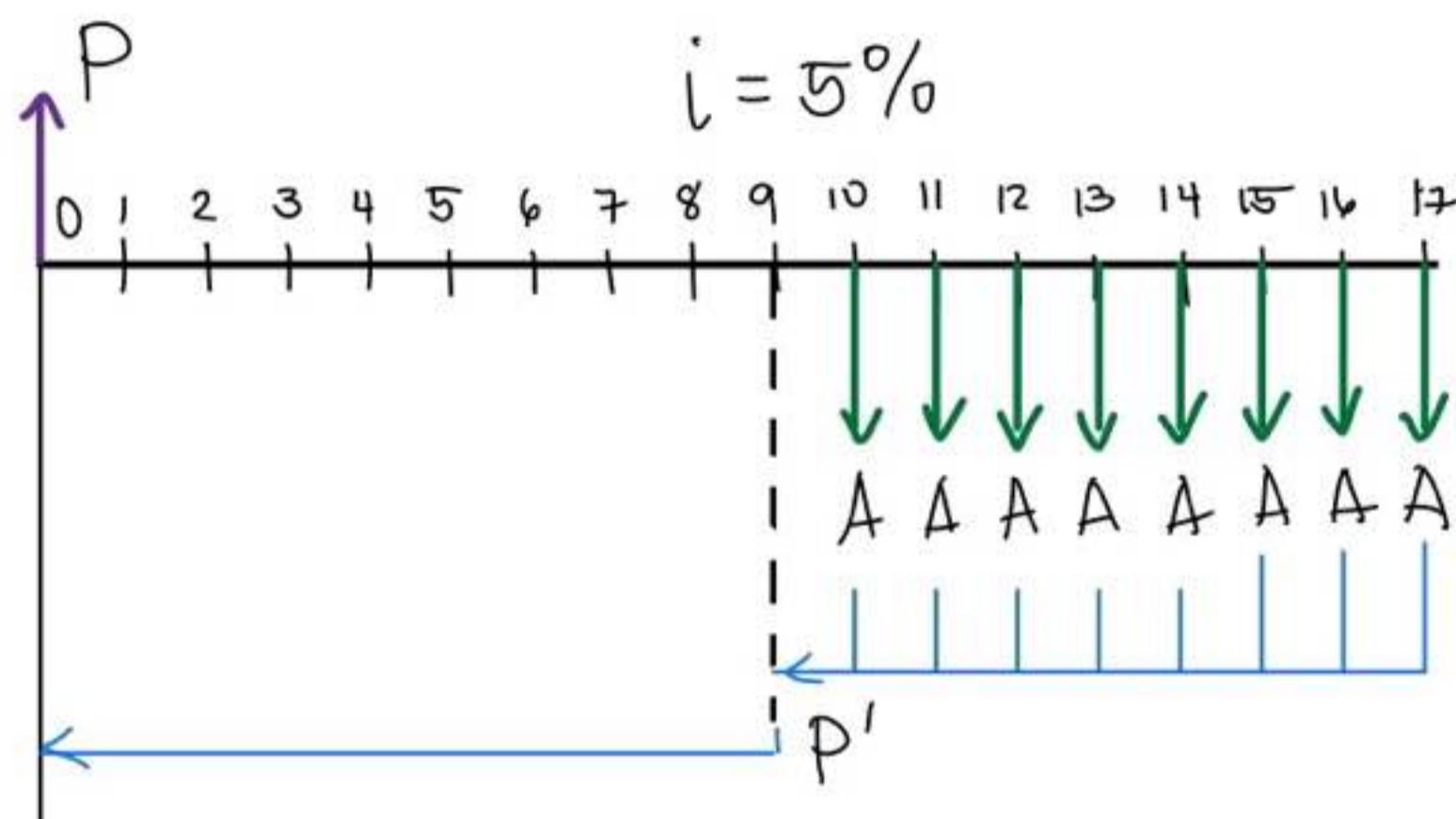
$$F = A \left[ \frac{(1+i)^n - 1}{i} \right]$$

Sample Problem 8

A man loans ₱187,400 from a bank with interest at 5% compounded annually. He agrees to pay his obligations by paying 8 equal annual payments the first being due at the end of 10 years. Find the annual payments.

Solution:

Given:  $P = ₱187,400$ ,  $i = 5\%$ ,  $n = 8$ ,  $k = 10$ ,  $A = ?$



$$P = A \left[ \frac{1 - (1+i)^{-n}}{i} \right] (1+i)^{-(k-1)}$$

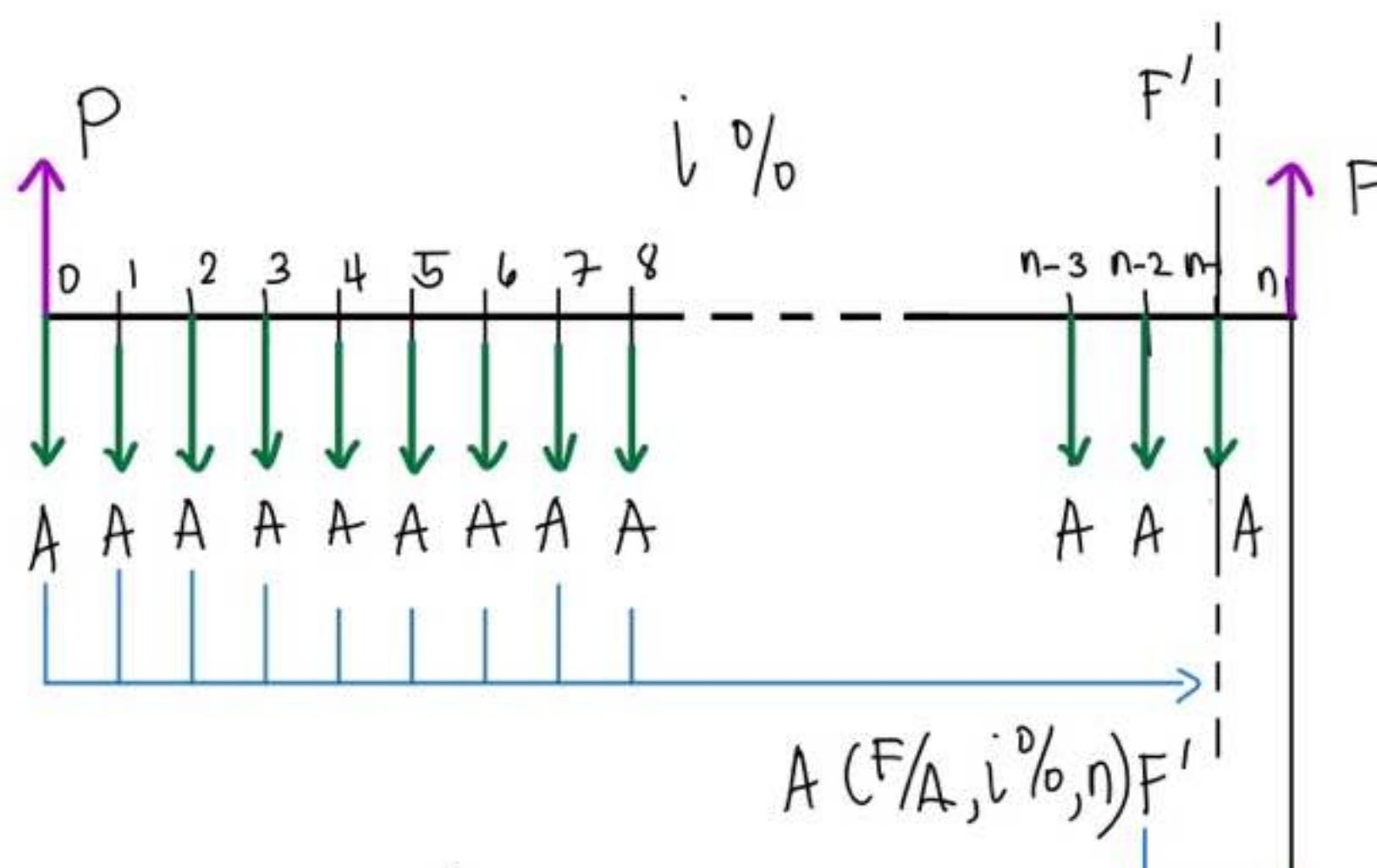
$$₱187,400 = A \left[ \frac{1 - (1+0.05)^{-8}}{0.05} \right] (1+0.05)^{-(10-1)}$$

Solving for  $A$ :

$$A = ₱44,980.56$$

### 3) Annuity Due

- the type of annuity where the payments are made at beginning of each period.



subscribe on youtube & fb  
@enginerdmath

$$F' = A \left[ \frac{(1+i)^n - 1}{i} \right]$$

$$F = F'(1+i)$$

$$F = A \left[ \frac{(1+i)^n - 1}{i} \right] (1+i)$$

$$= A \left[ \frac{(1+i)^{n+1} - (1+i)}{i} \right]$$

$$= A \left[ \frac{(1+i)^{n+1} - 1}{i} - \frac{i}{i} \right]$$

$$F = A \left[ \frac{(1+i)^{n+1} - 1}{i} - 1 \right]$$

$$P = A + A \left[ \frac{1 - (1+i)^{-(n-1)}}{i} \right]$$

$$P = A \left[ 1 + \frac{1 - (1+i)^{-(n-1)}}{i} \right]$$

### Sample Problem 9

A man bought an equipment costing ₱ 60,000 payable in 12 quarterly payments, each installment payable at the beginning of each period. The rate of interest is 24% compounded quarterly. What is the amount of each payment?

17

Solution: Given:  $P = \text{P}60,000$ ,  $r = 24\%$ ,  $m = 4$ ,  $n = 12$ ,  $A = ?$

$$P = A \left[ 1 + \frac{\left(1 + \frac{r}{m}\right)^{(nm-1)}}{\frac{r}{m}} \right]$$

subscribe on youtube & fb  
@enginerdmath

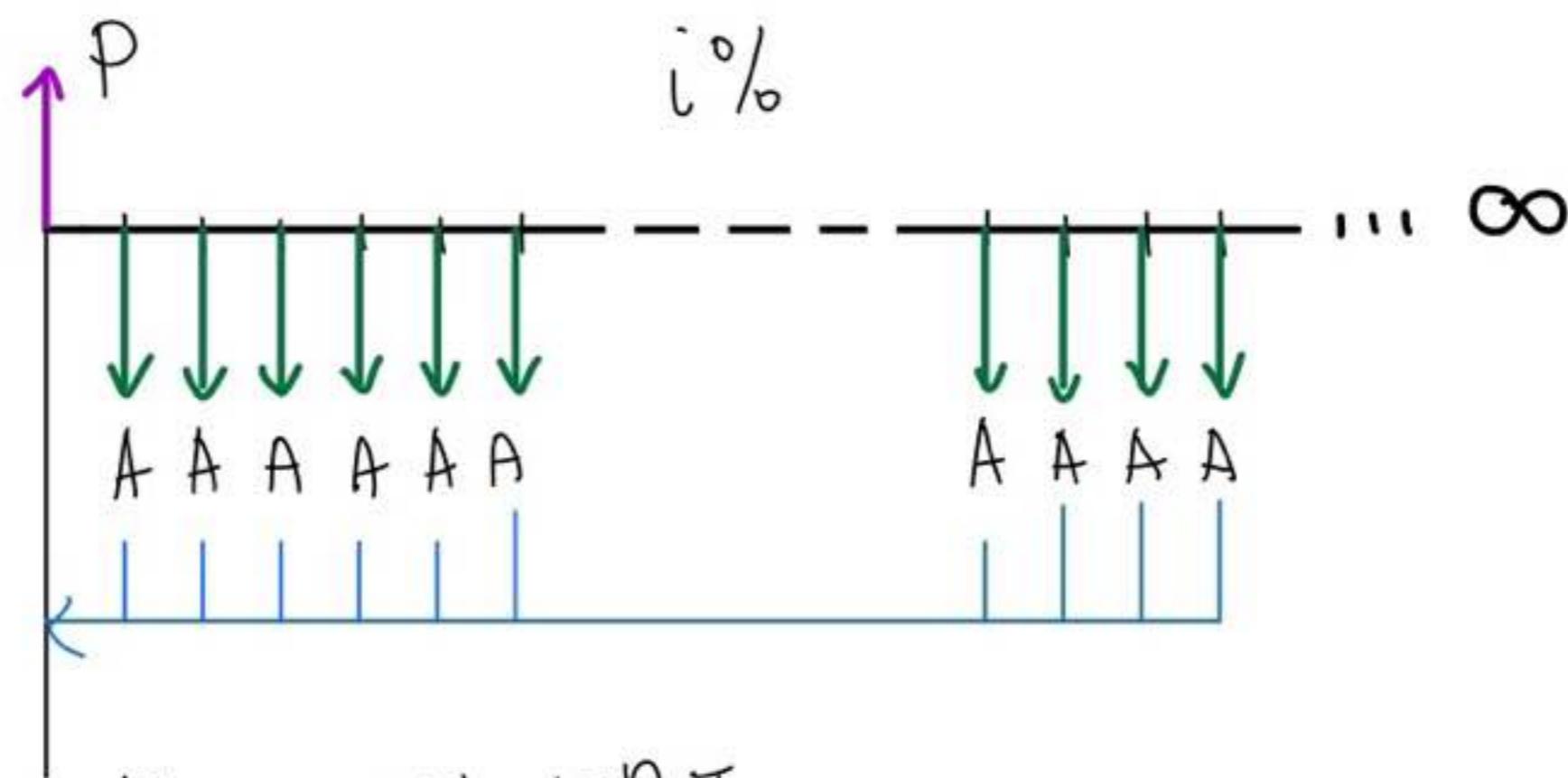
$$\text{P}60,000 = A \left[ 1 + \frac{\left(1 + \frac{0.24}{4}\right)^{-12-1}}{\frac{0.24}{4}} \right]$$

Solving for A:

$$A = \text{P}6751.52$$

#### 4) Perpetuity

- the type of annuity similar to ordinary annuity except that the payments continue infinitely.



$$P = A \left[ \frac{1 - (1+i)^{-n}}{i} \right]$$

but since  $n$  approaches infinity,  $(1+i)^{-n} \approx 0$

$$\therefore P = A \left[ \frac{1 - 0}{i} \right]$$

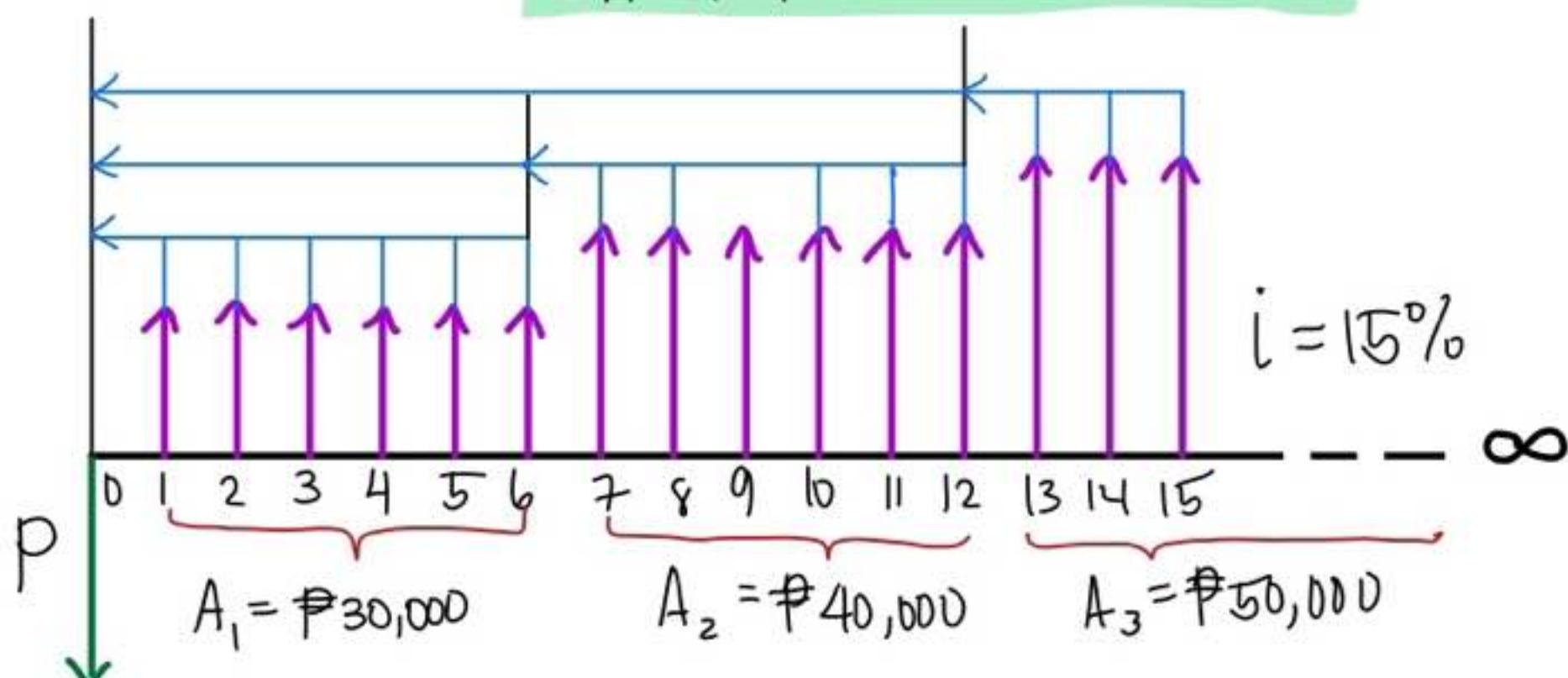
$$P = \frac{A}{i}$$

F cannot be determined.

**Sample Problem 10**

What amount of money invested today at 15% interest can provide the following scholarships: ₱ 30,000 at the end of each year for 6 years; ₱ 40,000 for the next 6 years and ₱ 50,000 thereafter.

Solution:

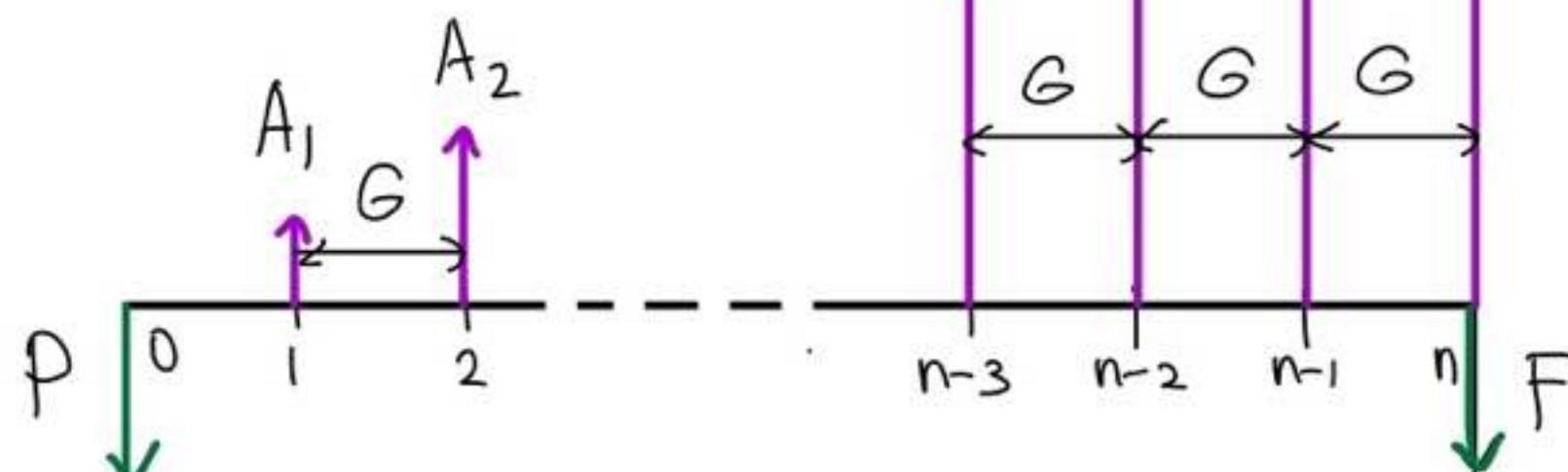
**CASH FLOW DIAGRAM**

$$P = ₱ 30,000 \left[ \frac{1 - (1 + 0.15)^{-6}}{0.15} \right] + ₱ 40,000 \left[ \frac{1 - (1 + 0.15)^{-6}}{0.15} \right] (1 + 0.15)^{-6} + \frac{₱ 50,000}{0.15} (1 + 0.15)^{-12}$$

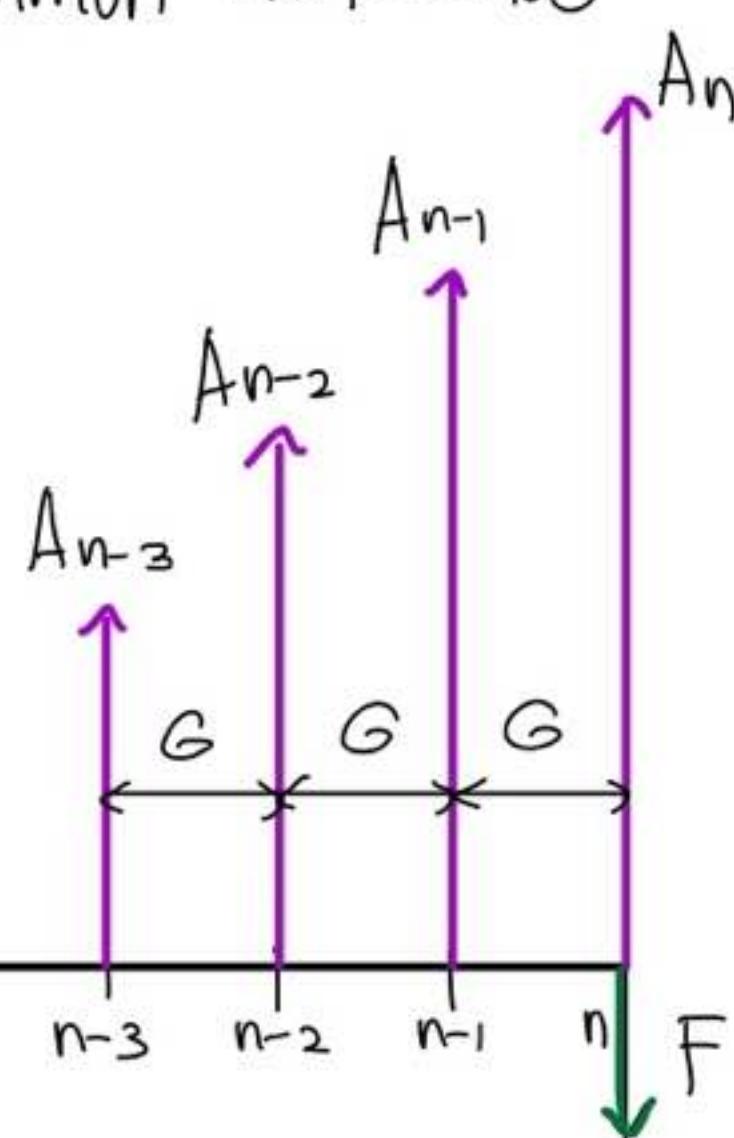
$$P = ₱ 241,282.32$$

**UNIFORM ARITHMETIC GRADIENT PAYMENT**

- a series of payments with common difference

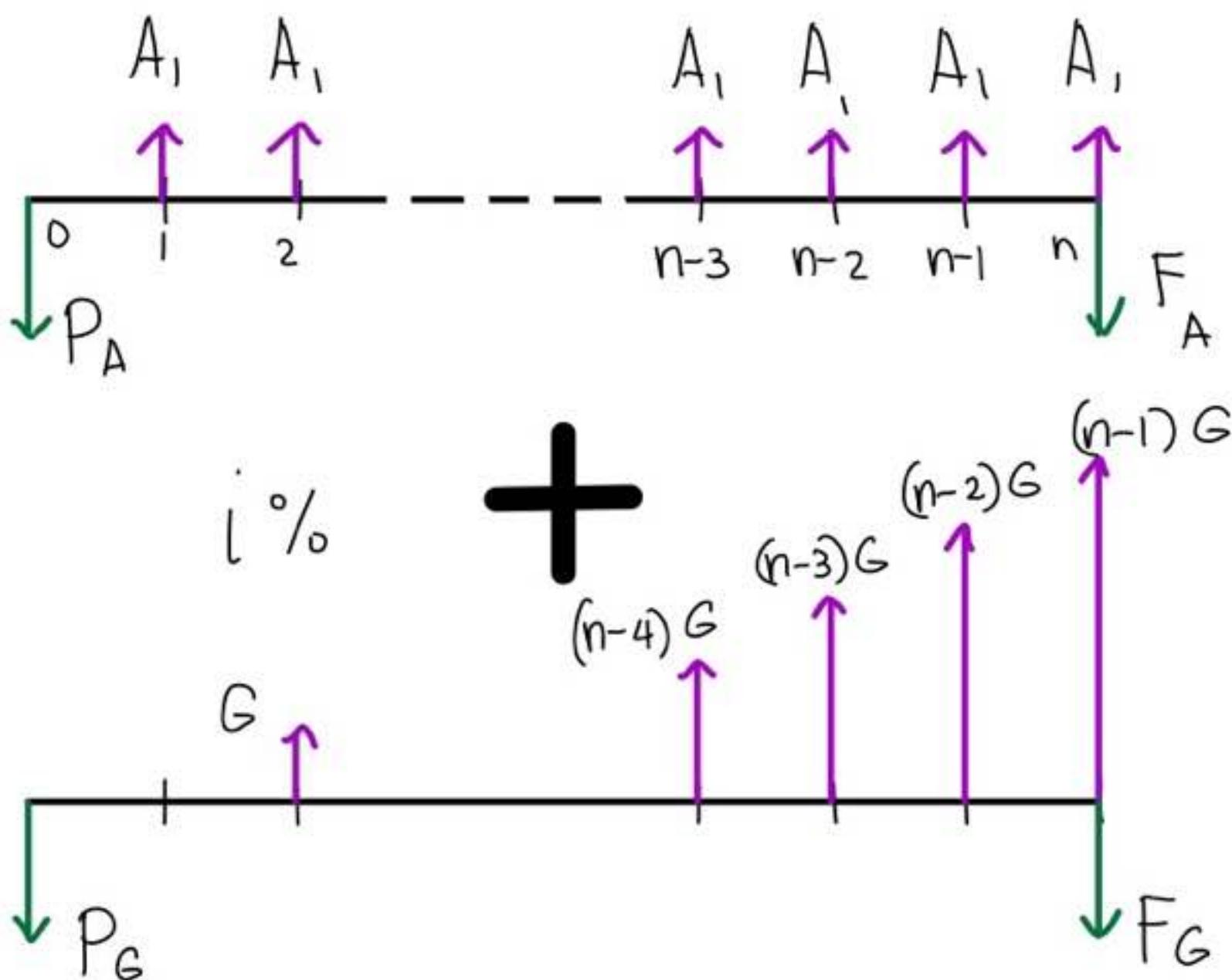
**For Ascending Payments****CASH FLOW DIAGRAM**

$$i\%$$



$$A_2 - A_1 = G$$

↓  
common difference



$$\therefore P = P_A + P_G$$

$$P_A = A_1 \left[ \frac{1 - (1+i)^{-n}}{i} \right]$$

$$P_G = \frac{G}{i} [1 - (1+i)^{-(n-1)}] (1+i)^{-1} + \frac{G}{i} [1 - (1+i)^{-(n-2)}] (1+i)^{-2}$$

$$\begin{aligned}
 &+ \dots + \frac{G}{i} [1 - (1+i)^{-1}] (1+i)^{-(n-1)} \\
 &= \frac{G}{i} [(1+i)^{-1} + (1+i)^{-2} + \dots + (1+i)^{-(n-1)}] - \frac{G}{i} (n-1)(1+i)^{-n} \\
 &= \frac{G}{i} [(1+i)^{-1} + (1+i)^{-2} + \dots + (1+i)^{-(n-1)}] + \frac{G}{i} (1+i)^{-n} - \frac{G}{i} n(1+i)^{-n} \\
 &= \frac{G}{i} [(1+i)^{-1} + (1+i)^{-2} + \dots + (1+i)^{-(n-1)} + (1+i)^{-n}] - \frac{G}{i} n(1+i)^{-n} \\
 &= \frac{G}{i} (1+i)^{-1} [1 + (1+i)^{-1} + \dots + (1+i)^{-(n-2)} + (1+i)^{-(n-1)}] - \frac{G}{i} n(1+i)^{-n}
 \end{aligned}$$

Using the geometric series formula:  $S_G = a_1 \left[ \frac{r^n - 1}{r - 1} \right]$

where  $a_1 = 1$  and  $r = (1+i)^{-1}$

$$P_G = \frac{G}{i} (1+i)^{-1} \left[ \left( 1 - \frac{(1+i)^{-n} - 1}{(1+i)^{-1} - 1} \right) - \frac{G}{i} n (1+i)^{-n} \right]$$

multiply the first part of the right side by  $\frac{1+i}{1+i}$

$$\begin{aligned} P_G &= \frac{G}{i} (1+i)^{-1} \left[ \frac{(1+i)^{-n} - 1}{(1+i)^{-1} - 1} \right] \frac{1+i}{1+i} - \frac{G}{i} n (1+i)^{-n} \\ &= \frac{G}{i} \left[ \frac{(1+i)^{-n} - 1}{1 - (1+i)} \right] - \frac{G}{i} n (1+i)^{-n} \\ &= \frac{G}{i} \left[ \frac{(1+i)^{-n} - 1}{-i} \right] - \frac{G}{i} n (1+i)^{-n} \\ &= \frac{G}{i} \left[ \frac{1 - (1+i)^{-n}}{i} \right] - \frac{G}{i} n (1+i)^{-n} \\ &\approx \frac{G}{i} \left[ \frac{1 - (1+i)^{-n}}{i} - n (1+i)^{-n} \right] \end{aligned}$$

$$P_G = \frac{G}{i} \left[ \frac{(1+i)^n - 1}{i} - n \right] (1+i)^{-n}$$

$$\therefore P_G = A_1 \left[ \frac{1 - (1+i)^{-n}}{i} \right] + \frac{G}{i} \left[ \frac{(1+i)^n - 1}{i} - n \right] (1+i)^{-n}$$

To get the equivalent annuity for this uniform arithmetic gradient payment

$$A_T \left[ \frac{1 - (1+i)^{-n}}{i} \right] = A_1 \left[ \frac{1 - (1+i)^n}{i} \right] + \frac{G}{i} \left[ \frac{(1+i)^{-1} - 1}{i} - n \right] (1+i)^{-n}$$

$$A_T \frac{(1+i)^n - 1}{i (1+i)^n} = A_1 \frac{(1+i)^n - 1}{i (1+i)^n} + \frac{G}{i} \left[ \frac{(1+i)^n - 1}{i} - n \right] (1+i)^{-n}$$

21

$$\begin{aligned}
 A_T &= A_1 + \frac{G}{i} \left[ \frac{(1+i)^n - 1}{i} - n \right] (1+i)^{-n} \frac{i(1+i)^n}{(1+i)^n - 1} \\
 &= A_1 + \frac{G}{i} \left[ \frac{(1+i)^n - 1}{i[(1+i)^n - 1]} - \frac{ni}{(1+i)^n - 1} \right]
 \end{aligned}$$

subscribe on youtube & fb  
@engineerdmath

$$A_T = A_1 + \frac{G}{i} \left[ 1 - \frac{ni}{(1+i)^n - 1} \right]$$

To find F:

$$F = F_A + F_G$$

$$F_A = A_1 \left[ \frac{(1+i)^n - 1}{i} \right]$$

$$\begin{aligned}
 F_G &= \frac{G}{i} \left[ (1+i)^{n-1} - 1 \right] + \frac{G}{i} \left[ (1+i)^{n-2} - 1 \right] + \dots + \frac{G}{i} \left[ (1+i) - 1 \right] \\
 &= \frac{G}{i} \left[ (1+i)^{n-1} + (1+i)^{n-2} + \dots + (1+i) \right] - \frac{G}{i} (n-1)
 \end{aligned}$$

$$= \frac{G}{i} \left[ (1+i)^{n-1} + (1+i)^{n-2} + \dots + (1+i) + 1 \right] - \frac{G}{i} n$$

$$\text{Using } S_G = a_1 \frac{r^{n-1}}{r-1} \quad \text{where } a_1 = 1, r = (1+i)$$

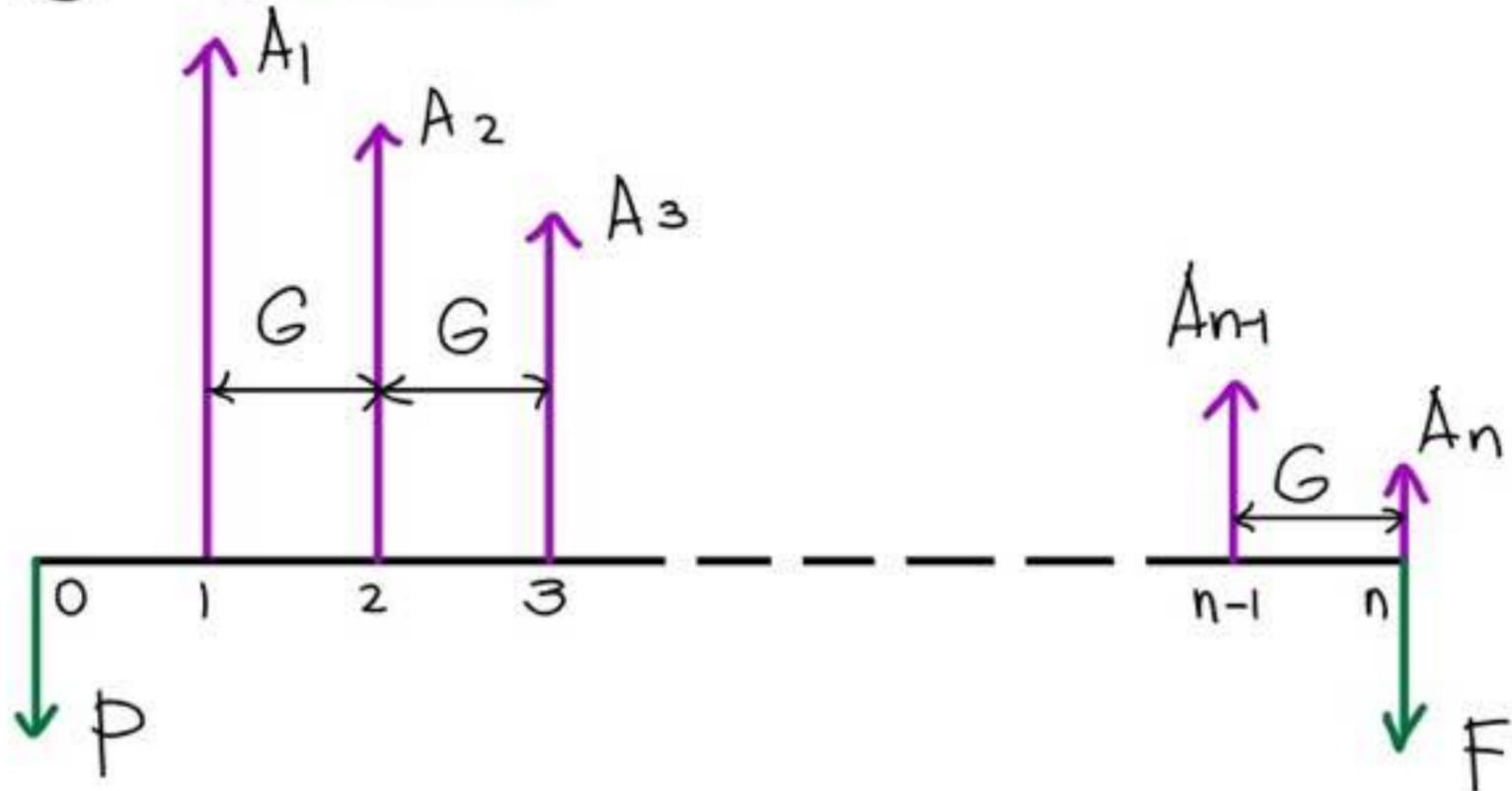
$$F_G = \frac{G}{i} \left[ (1) \frac{(1+i)^{n-1}}{(1+i) - 1} \right] - \frac{G}{i} n$$

$$= \frac{G}{i} \left[ \frac{(1+i)^n - 1}{(1+i) - 1} \right] - \frac{G}{i} n$$

$$F_G = \frac{G}{i} \left[ \frac{(1+i)^n - 1}{i} - n \right]$$

$$\therefore F = A_i \left[ \frac{(1+i)^n - 1}{i} \right] + \frac{G}{i} \left[ \frac{(1+i)^n - 1}{i} - n \right]$$

### For Descending Payments



$$P = P_{A_1} - P_G$$

$$P = A_i \left[ \frac{1 - (1+i)^{-n}}{i} \right] - \frac{G}{i} \left[ \frac{(1+i)^n - 1}{i} - n \right] (1+i)^{-n}$$

$$F = F_{A_1} - F_G$$

$$F = A_i \left[ \frac{(1+i)^n - 1}{i} \right] - \frac{G}{i} \left[ \frac{(1+i)^n - 1}{i} - n \right]$$

$$A_T = A_i - \frac{G}{i} \left[ 1 - \frac{n}{(1+i)^n - 1} \right]$$

### Sample Problem 11

Find the equivalent annual payment of the following obligations at 20% interest.

Interest	End of year	Payment
	1	₱ 8,000
	2	₱ 7,000
	3	₱ 6,000
	4	₱ 5,000

23

subscribe on youtube & fb  
@engineerdmath

Solution:

Given:  $A_1 = \text{P}8000$ ;  $G = \text{P}1000$ ;  $n = 4$ ,  $i = 20\%$ ,  $A_T = ?$ 

Since this is a descending payment, we will use the formula above:

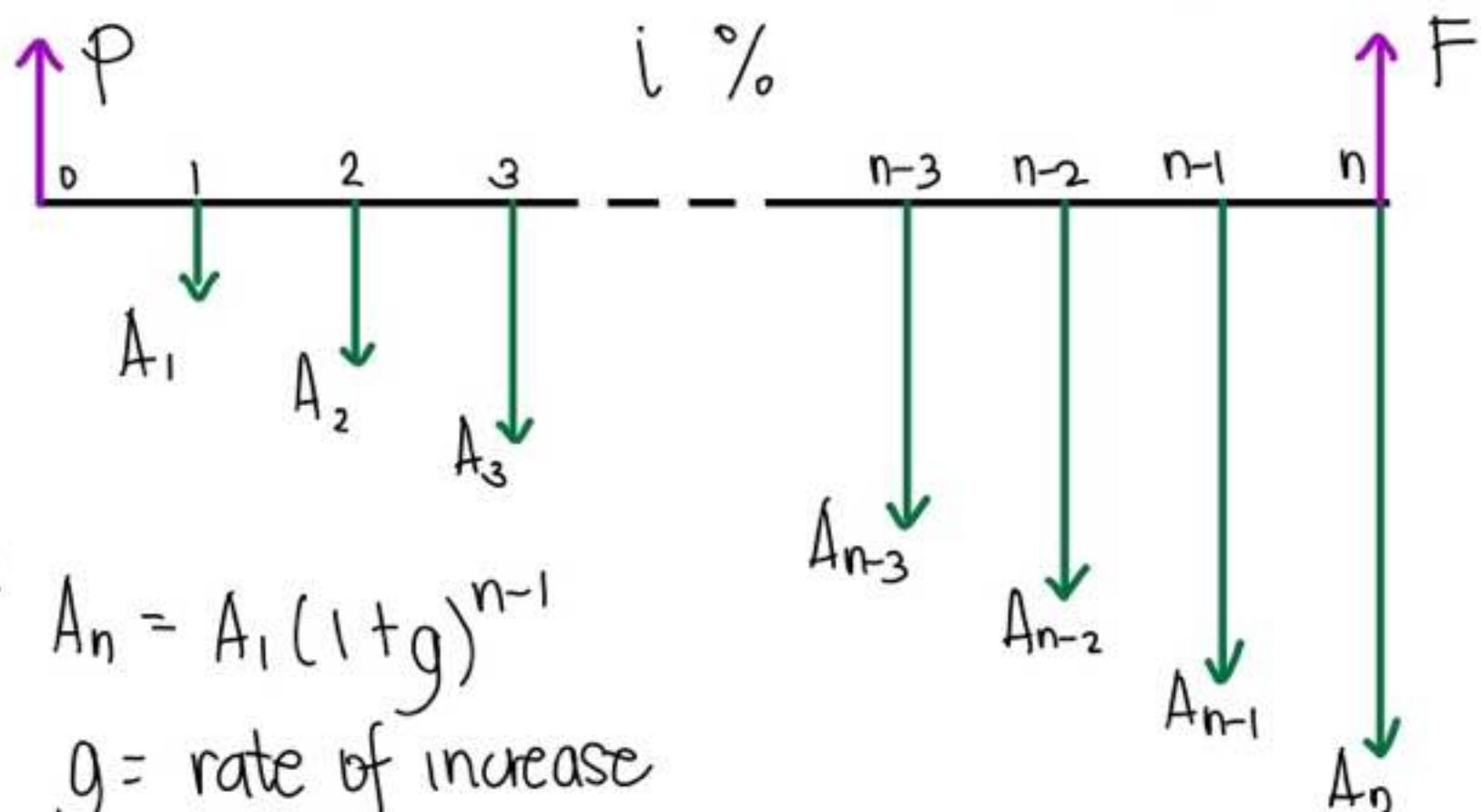
$$A_T = A_1 - \frac{G}{i} \left[ 1 - \frac{ni}{(1+i)^n - 1} \right]$$

$$A_T = \text{P}8000 - \frac{\text{P}1000}{0.2} \left[ 1 - \frac{4(0.20)}{(1+0.2)^4 - 1} \right]$$

$$A_T = 6725.78$$

## GEOMETRIC GRADIENT

### CASH FLOW DIAGRAM



where:

$$A_n = A_1 (1+g)^{n-1}$$

$g$  = rate of increase

Using 0 as Focal date:

$$P = A_1 (1+i)^{-1} + A_2 (1+i)^{-2} + A_3 (1+i)^{-3} + \dots + A_n (1+i)^{-n}$$

$$A_n = A_1 (1+g)^{n-1}$$

$$P = \sum_{j=1}^n A_1 (1+g)^{n-1}$$

$$= \sum_{j=1}^n A_1 (1+g)^{n-1} (1+i)^{-n}$$

$$= \sum_{j=1}^n \frac{A_1}{1+g} \left[ \frac{1+i}{1+g} \right]^{-n}$$

24

subscribe on youtube & fb  
 @enginerdmath

$$\text{let } \frac{1+i}{1+g} = 1 + i_{cr}$$

$$i_{cr} = \frac{1+i}{1+g} - 1$$

$$P = \sum_{j=1}^n \frac{A_1}{1+g} [1 + i_{cr}]^{-j}$$

$$= \frac{A_1}{1+g} \sum_{j=1}^n [1 + i_{cr}]^{-j} \xrightarrow{\frac{1 - (1+i)^{-n}}{i}} \text{ where } i = i_{cr}$$

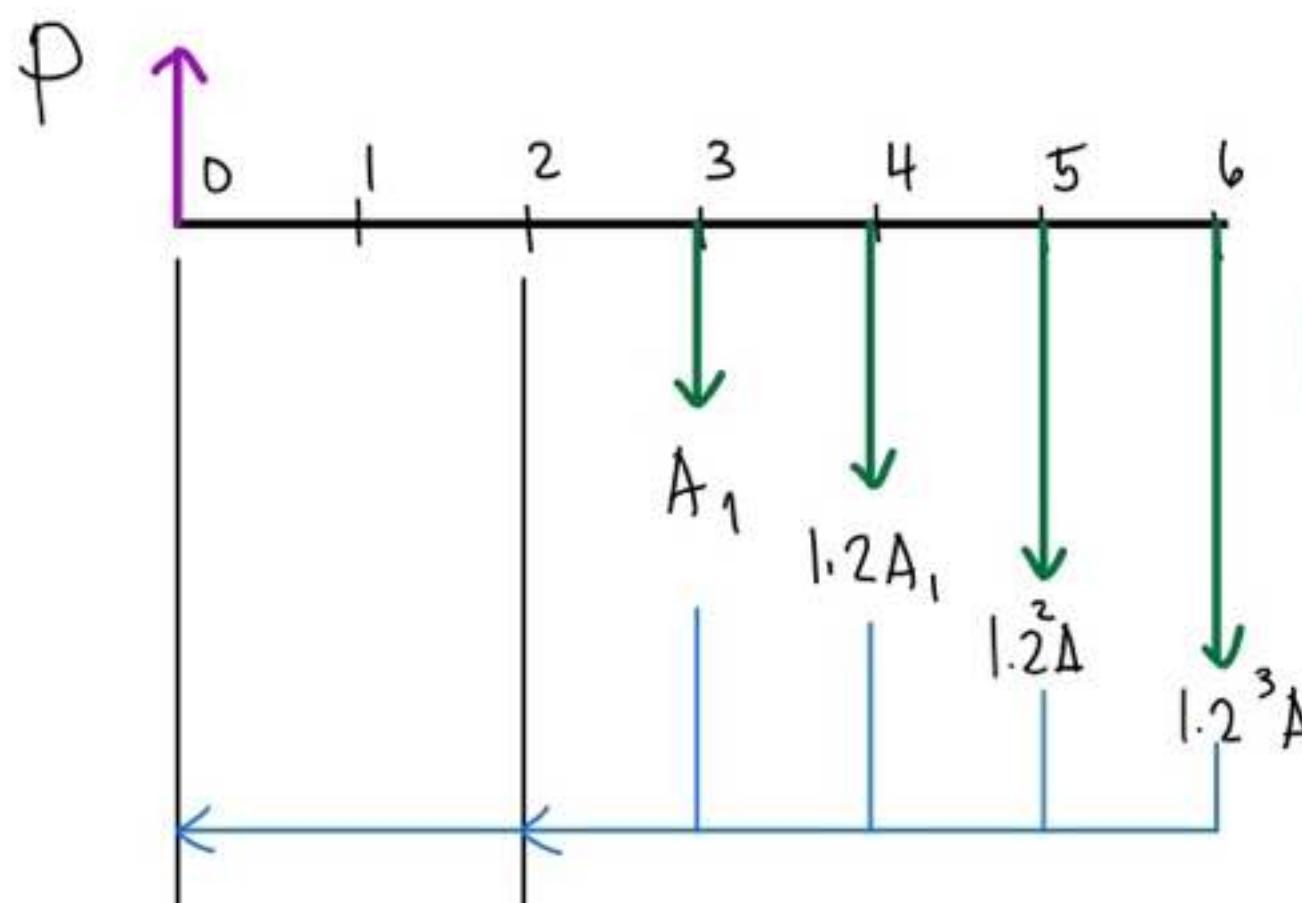
$$P = \frac{A_1}{1+g} \left[ \frac{1 - (1 + i_{cr})^{-n}}{i_{cr}} \right]$$

### Sample Problem 12

A ₱1M debt is to be paid in 4 installments, the next payment being 20% larger than the preceding. If money is worth 10% and the first payment is made 3 years after the debt has been granted. Compute the first payment during the third year.

Solution: This is an example of geometric gradient with deferral.

Given:  $P = ₱1M$ ,  $n = 4$ ,  $i = 10\%$ ,  $g = 20\%$ ,  $A_1 = ?$



CASH FLOW DIAGRAM

25

subscribe on youtube & fb  
@engineerdmath

From the cash flow diagram, setting D as focal date:

$$P = \frac{A_1}{1+g} \left[ \frac{1 - (1+i_{cr})^{-n}}{i_{cr}} \right] (1+i)^{-2} \quad \text{where } i_{cr} = \frac{1+i}{1+g} - 1$$

$$\therefore P_{1000,000} = \frac{A_1}{1+0.2} \left[ \frac{1 - \left(1 + \frac{1+0.1}{1+0.2} - 1\right)^{-4}}{\frac{1+0.1}{1+0.2} - 1} \right] (1+0.1)^{-2}$$

Solving for  $A_1$ :

$$A_1 = P 290,658.08$$

## Continuous Compounding and Discrete Cash Flows

$$F = P(1+i)^{my}, \quad m \rightarrow \infty$$

$$\text{let } x = \frac{m}{r} \quad \therefore m = xr$$

$$F = P \left(1 + \frac{r}{m}\right)^{my}$$

$$= P \left(1 + \frac{1}{x}\right)^{xry}$$

$$F = P \left[ \left(1 + \frac{1}{x}\right)^x \right]^{ry}$$

$$\text{but } \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e$$

$$\therefore F = Pe^{ry}$$

$$1+i = e^r$$

$$i = e^r - 1$$

$$P = Fe^{-ry}$$

26

$$F = A \left[ \frac{(1+i)^{my} - 1}{i} \right] \rightarrow F = A \left[ \frac{e^{ry} - 1}{e^r - 1} \right]$$

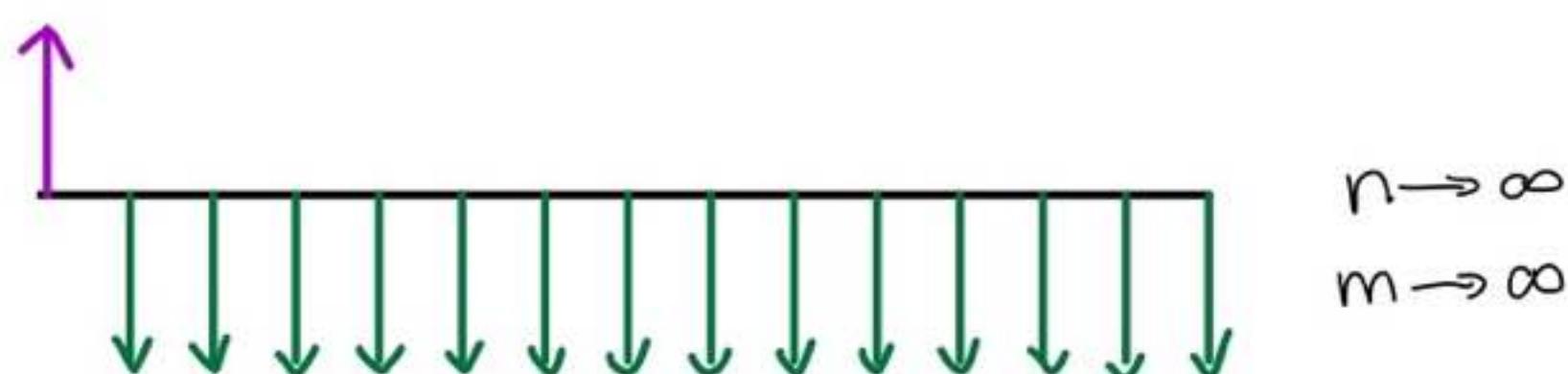
$$A = \frac{F(e^r - 1)}{e^{ry} - 1}$$

subscribe on youtube & fb  
@enginerdmath

$$P = A \left[ \frac{1 - (1+i)^{-my}}{i} \right] \rightarrow P = A \left[ \frac{1 - e^{-ry}}{e^r - 1} \right]$$

$$A = \frac{P(e^r - 1)}{1 - e^{-ry}}$$

## Continuous Compounding and Continuous Cash Flow



$$F = A \left[ \frac{e^m - 1}{r} \right]$$

$$A = F \left[ \frac{r}{e^m - 1} \right]$$

$$P = A \left[ \frac{1 - e^{-rn}}{r} \right] = A \left[ \frac{e^{rn} - 1}{r e^{rn}} \right]$$

$$A = \frac{Pr}{1 - e^{-rn}} = \frac{Pr e^{rn}}{e^{rn} - 1}$$

## Sample Problem 1B

Find the accumulated amount of money after 5 years of ₱1000 invested at the rate of 10% per year compounded continuously.

Solution: Given:  $P = ₦1000$ ,  $r = 10\%$ ,  $n = y = 5$  @engineerdmath

$$\therefore F = P e^{ry} = ₦1000 e^{(0.1)(5)} = ₦1,648.72$$

## DEPRECIATION

- the decrease in value of physical properties with the passage of time
- ex: properties that depreciates: car, computer, refrigerator, etc.

### 1) Straight Line Method (SLM)

- it assumes uniform annual depreciation. It is the most extensively used method of depreciation because of its simplicity.

$$d = \frac{C_0 - C_L}{L}$$

where:

$L$  = useful life of the property in years

$C_0$  = the original cost

$C_L$  = the value at the end of the life, the scrap value (including gain or loss due to removal)

$d$  = annual cost of depreciation

$C_n$  = the book value at the end of  $n$  years

$D_n$  = depreciation up to age  $n$  years

$$D_n = n \left[ \frac{C_0 - C_L}{L} \right]$$

### Sample Problem 14

A machine has an initial cost of ₦50,000 and a salvage value of ₦10,000 after 10 years. What is the straight line method depreciation rate as a percentage of the initial cost?

Solution:

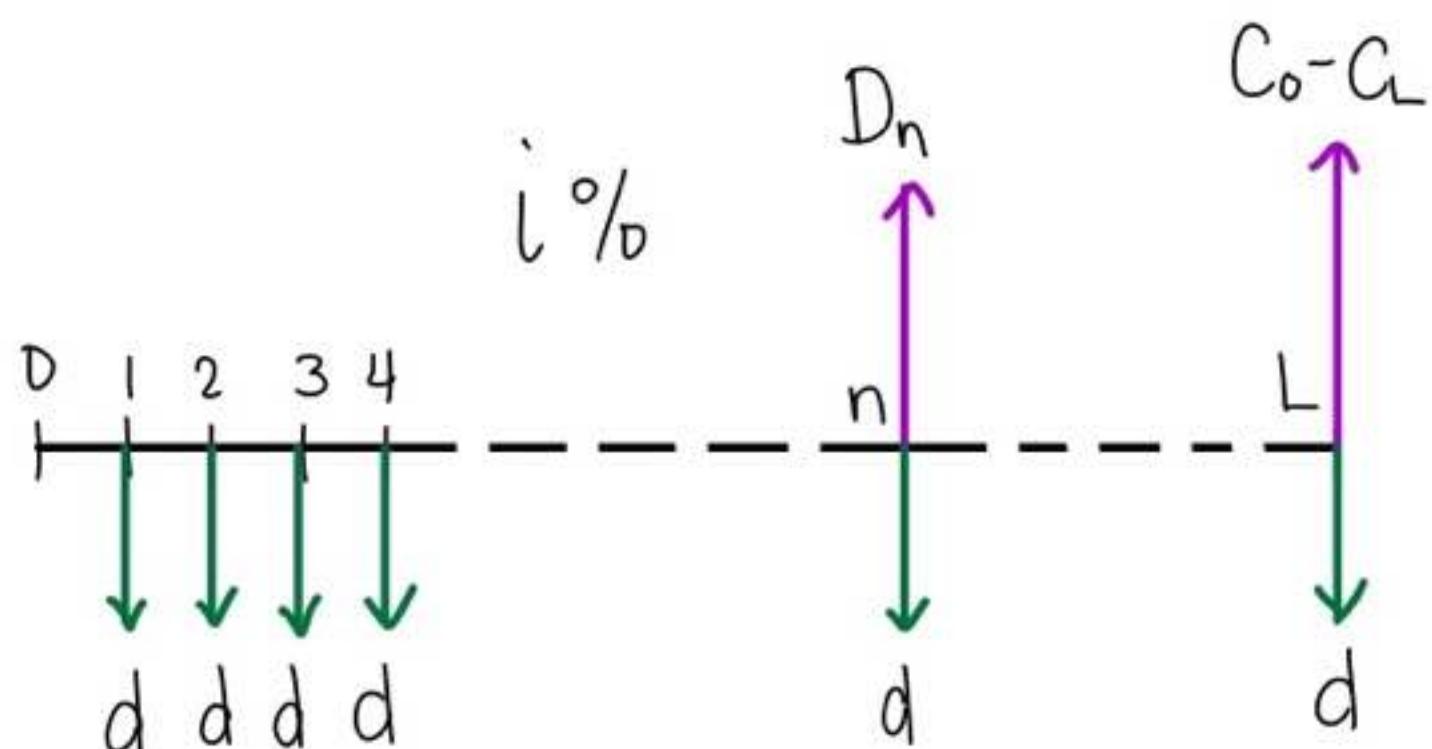
Given:  $C_0 = \text{P}50,000$ ;  $C_L = \text{P}10,000$ ,  $L = 10$ ,  $\frac{d}{C_0} = ?$

$$d = \frac{C_0 - C_L}{L} = \frac{\text{P}50,000 - \text{P}10,000}{10} = \text{P}4000$$

$$\frac{d}{C_0} = \frac{\text{P}4000}{\text{P}50,000} = 0.08 \text{ or } 80\%$$

## 2) Sinking Fund Method (SFM)

- It assumes uniform annual depreciation deposited in a sinking fund whose accumulation is equal to the total depreciation up to the period considered.



Using the formula for the future cost of annuity where:  $A = d$  and  $F = C_0 - C_L$

$$F = A \left[ \frac{(1+i)^n - 1}{i} \right]$$

$$C_0 - C_L = d \left[ \frac{(1+i)^L - 1}{i} \right]$$

$$d = \frac{(C_0 - C_L) i}{(1+i)^L - 1}$$

Also,  $d = \frac{C_0 - C_n}{(1+i)^n - 1}$

$$C_n = C_0 - D_n$$

$$D_n = d \left[ \frac{(1+i)^n - 1}{i} \right]$$

Sample Problem 15

1) A unit welding machine costs ₦45,000 with an estimated life of 5 years. Its salvage value is ₦2,500, find its depreciation using sinking fund method at 8.5%

Solution: Given:  $L=5$ ;  $C_0 = ₦45,000$ ;  $C_L = ₦2,500$ ;  $i = 8.5\%$ ,  $d = ?$

$$d = \frac{(C_0 - C_L)i}{(1+i)^L - 1} = \frac{(₦45,000 - ₦2,500)(0.085)}{(1 + 0.085)^5 - 1} = ₦7172.54$$

3) Declining Balance Method (DBM) or Matheson Formula Method  
or Fixed / Constant Percentage Method

- this method takes depreciation charge during the year as a fixed or constant percentage of the book value at the beginning of the year.

$$d_n = C_0 (1 - k)^{n-1} k$$

$$C_n = C_0 (1 - k)^n = C_0 \left[ \frac{C_L}{C_0} \right]^{\frac{n}{L}}$$

$$C_L = C_0 (1 - k)^L$$

where:

$k$  = rate of depreciation

$$k = 1 - \sqrt[n]{\frac{C_n}{C_0}} = 1 - \sqrt[L]{\frac{C_L}{C_0}}$$

## 4) Double Declining Balance Method (DDBM)

- similar to the declining balance method except that  $k = \frac{2}{L}$

$$d_n = \frac{2C_0}{L} \left[ 1 - \frac{2}{L} \right]^{n-1}$$

$$C_n = C_0 \left[ 1 - \frac{2}{L} \right]^n$$

## Sample Problem 1b

A machine having a first cost of ₦ 60,000 will be retired at the end of 8 years. Depreciation cost is computed using constant percentage method. What is the total cost of depreciation up to the time the machine is retired if the annual rate of depreciation is 28.72%?

Solution: Given:  $C_0 = ₦ 60,000$ ;  $L = 8$ ;  $k = 28.72\%$ ;  $D_8 = ?$

$$C_L = C_0 (1 - k)^L = ₦ 60,000 (1 - 0.2872)^8 = ₦ 3998.46$$

$$D_8 = C_0 - C_8 = ₦ 60,000 - ₦ 3998.46 = ₦ 56,001.54$$

## 5) Sum-of-the Years Digit Method (SYDM)

- the depreciation charge is computed based on the reverse digit and sum of the digits

$$d_n = \frac{\text{Reverse Digit}}{\text{Sum of Digits}} (C_0 - C_L)$$

$$d_n = \frac{2}{L(L+1)} \frac{L-m+1}{m} (C_0 - C_L)$$

3)

subscribe on youtube & fb  
 @enginerdmath

$$C_n = C_0 - D_n$$

$$D_n = d_1 + d_2 + d_3 + \dots + d_n$$

$$D_n = \frac{m(2L-m+1)}{L(L+1)} (C_0 - C_L)$$

### Sample Problem 17

An asset is purchased for ₦9,000. Its estimated life is 10 years after which it will be sold for ₦1,000. Find the book value during the second year if sum-of-the-years digits (SYDM) depreciation is used.

Solution: Given:  $C_0 = ₦9,000$ ;  $C_L = ₦1,000$ ,  $L = 10$ ,  $m = 2$ ,  $C_2 = ?$

$$C_n = C_0 - D_n$$

$$D_n = \frac{m(2L-m+1)}{L(L+1)} [C_0 - C_L]$$

$$D_2 = \frac{2(2(10)-2+1)}{10(10+1)} [₦9,000 - ₦1,000]$$

$$D_2 = ₦2763.64$$

$$\therefore C_2 = ₦9,000 - ₦2763.64$$

$$= ₦6236.36$$

## 6) Service Output Method (SOM)

-the method of depreciation that assumes the depreciation charge is directly proportional to the number of units produced or number of hours in service.

- based on output units

$$d_n = \left[ \frac{C_0 - C_L}{T} \right] (Q_n)$$

$$C_n = C_0 - D_n$$

$$D_n = d_1 + d_2 + d_3 + \dots + d_n$$

where:  $T$  = total units of output up to the end life

$Q_n$  = total number of units of output during the  $n$ th year

- based on hours in service

$$d_n = \left[ \frac{C_0 - C_L}{H_T} \right] (H_n)$$

$$C_n = C_0 - D_n$$

$$D_n = d_1 + d_2 + d_3 + \dots + d_n$$

where:  $H_T$  = total number of hours up to the end life

$H_n$  = total number of hours during the  $n$ th year

### Sample Problem 18

A Television Company purchased machinery for ₱ 100,000 on July 1, 1979. It is estimated that it will have a useful life of 10 years; scrap value of ₱ 4,000, production of 400,000 units and working hours of 120,000.

The company uses the machinery for 14,000 hours in 1979 and 18,000 hours in 1980. The machinery produces 36,000 units in 1979 and 44,000 units in 1980. Compute the depreciation for 1980 using service output method.

Solution: Given:  $C_0 = \text{P} 100,000$ ;  $C_L = \text{P} 4,000$ ;  $L = 10 \text{ years}$

$T = 400,000 \text{ units}$ ;  $H_T = 120,000 \text{ hours}$

$Q_{80} = 44,000 \text{ units}$ ;  $H_{80} = 18,000 \text{ hours}$

Based on output units:

$$d_{80} = \left[ \frac{C_0 - C_L}{T} \right] (Q_{80}) = \left[ \frac{\text{P} 10,000 - \text{P} 4,000}{400,000} \right] (44,000)$$

$$= \boxed{\text{P} 10,560}$$

Based on hours in service:

$$d_{80} = \left[ \frac{C_0 - C_L}{H_T} \right] (H_{80}) = \left[ \frac{\text{P} 10,000 - \text{P} 4,000}{120,000} \right] (18,000)$$

$$= \boxed{\text{P} 14,400}$$

## DEPLETION

- is the decrease in the value of a certain property or natural resources such as mines, oil, timber, quarries, etc. due to gradual extraction of its contents.

### Methods of Computing Depletion Charge Per Year

#### I) Percentage or Depletion Allowance Method

Depletion Cost during the year = Fixed Percentage of Gross Annual Income

## 2) Unit Used Factor Method or Cost Method

Depletion cost during the year =  $\left[ (\text{initial cost of property}) (\text{units sold during the year}) \right] / (\text{total units in the property})$

## INFLATION

- it is the decrease in the value of a currency due to the increase in volume of currency circulated that will eventually increase the price of the commodities

CPI Annual Inflation Rate for the Year =  $(\text{CPI for the year} - \text{CPI from the previous year}) / \text{CPI for the previous year}$

CPI = Consumer Price Index

$$F = P(1+f)^n$$

→ inflation equation

OR

$$F = P(1+i_c)^n$$

where:

$i_c$  = combined interest rate

$i_r$  = real interest rate

$f$  = average inflation rate per year

$$i + i_c = (1+f)(1+i_r)$$

## Sample Problem 19

A machine costs ₱20,000 today. If inflation is 6% per year and interest is 10% per year, what will be the approximate future worth of the machine, adjusted for inflation, in 5 years?

Solution: Given:  $P = \text{P} 20,000$ ;  $f = 6\%$ ;  $i_r = 10\%$ ,  $n = 5$ ,  $F = ?$

$$F = P(1+i_c)^n \quad \text{where } 1+i_c = (1+f)(1+i_r)$$

$$\begin{aligned} F &= P[(1+f)(1+i_r)]^n \\ &= \text{P} 20,000 [(1+0.06)(1+0.1)]^5 \\ &= \boxed{\text{P} 43,104.51} \end{aligned}$$

subscribe on youtube & fb  
@enginerdmath

## CAPITALIZED COST

-the sum of the first cost and the present worth of all cost of replacement operation and maintenance for a long time or forever.

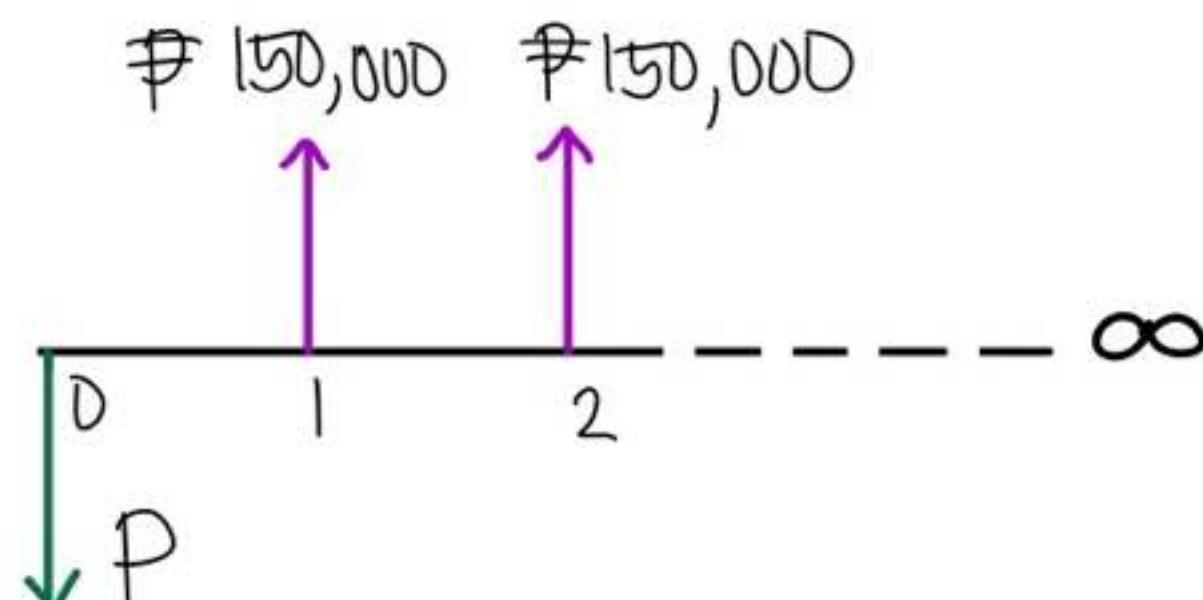
Case 1: No replacement, only maintenance and/or operation every period

Capitalized Cost = First cost + Present Worth of perpetual operation and/or maintenance.

### Sample Problem 20:

Determine the capitalized cost of a structure that requires an initial investment of  $\text{P} 1,500,000$  and an annual maintenance of  $\text{P} 150,000$ . Interest is  $15\%$ .

Solution:



CASH FLOW DIAGRAM

3b

$$P = \frac{A}{i} = \frac{\text{P} 150,000}{0.15} = \text{P} 1,000,000$$

subscribe on youtube & fb  
@enginerdmath

$$\text{Capitalized Cost} = \text{First cost} + P$$

$$= \text{P} 1,500,000 + \text{P} 1,000,000$$

$$= \boxed{\text{P} 2,500,000}$$

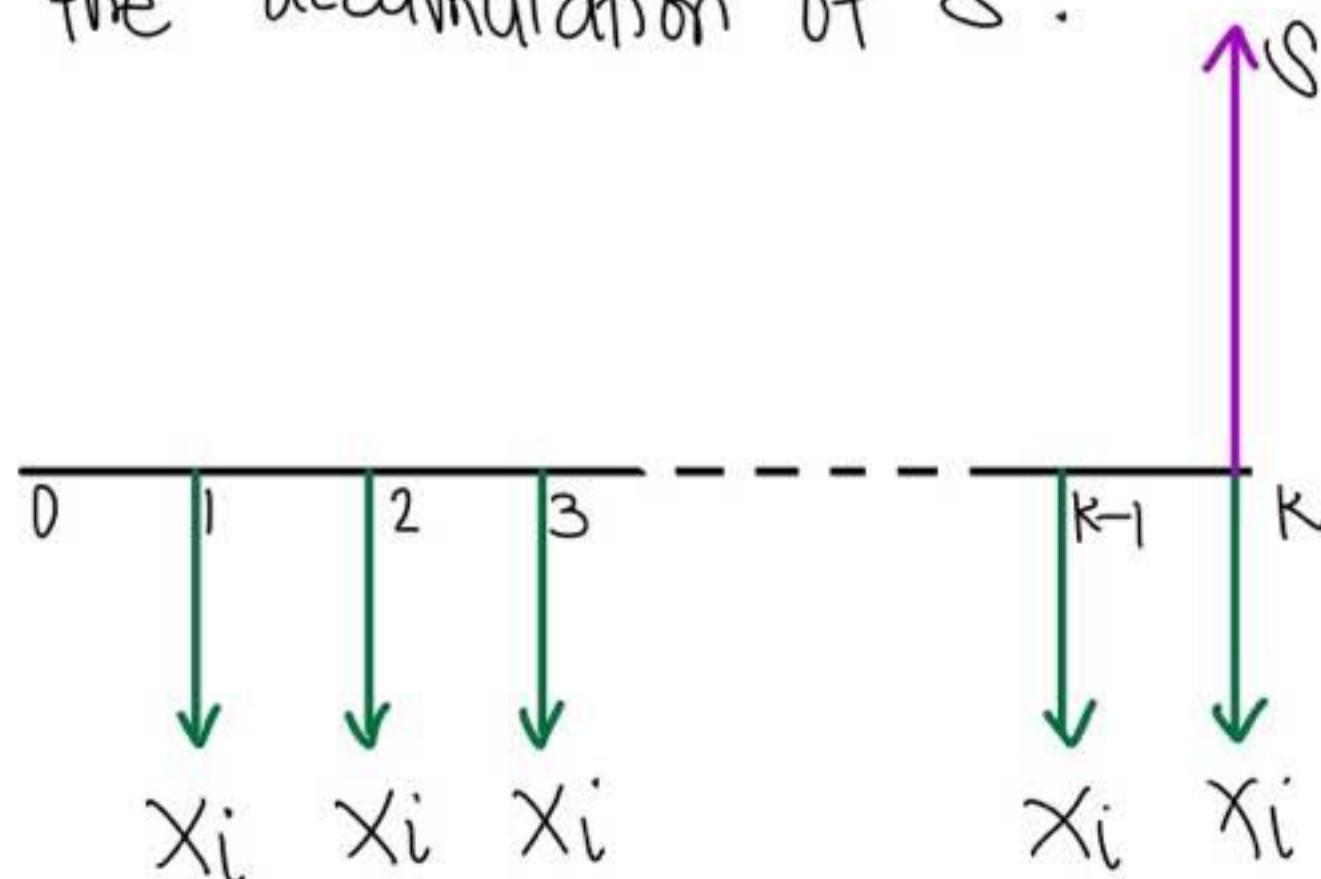
Case 2: Replacement only, no maintenance and or operation

$$\text{Capitalized cost} = \text{First cost} + \text{Present worth of perpetual replacement}$$

Let  $S$  = amount needed to replace a property every  $k$  periods

$X$  = amount of principal invested at  $i\%$  the interest on which will amount to  $S$  every  $k$  periods.

$X_i$  = interest on  $X$  every period, the period deposit towards the accumulation of  $S$ .

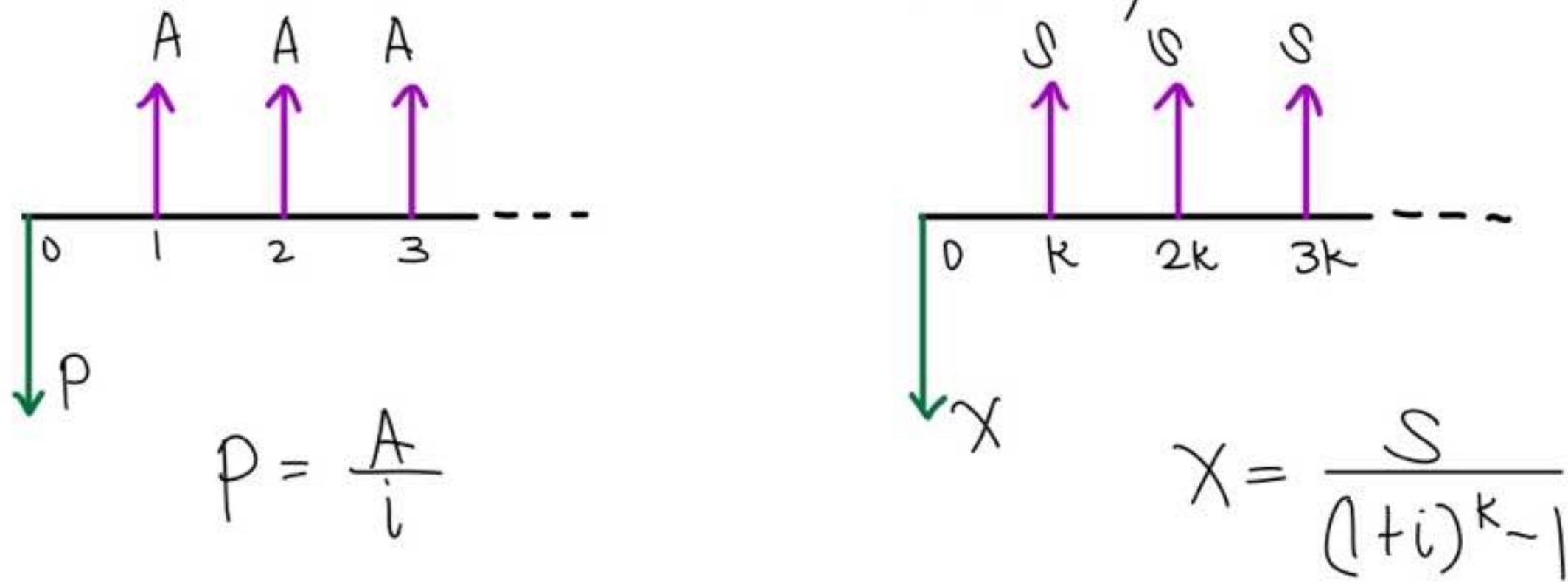


$$S = X_i \left[ \frac{(1+i)^k - 1}{i} \right]$$

$$X_i = \frac{S}{(1+i)^k - 1}$$

Difference between P and X in a perpetuity

subscribe on youtube & fb  
@engineerdmath



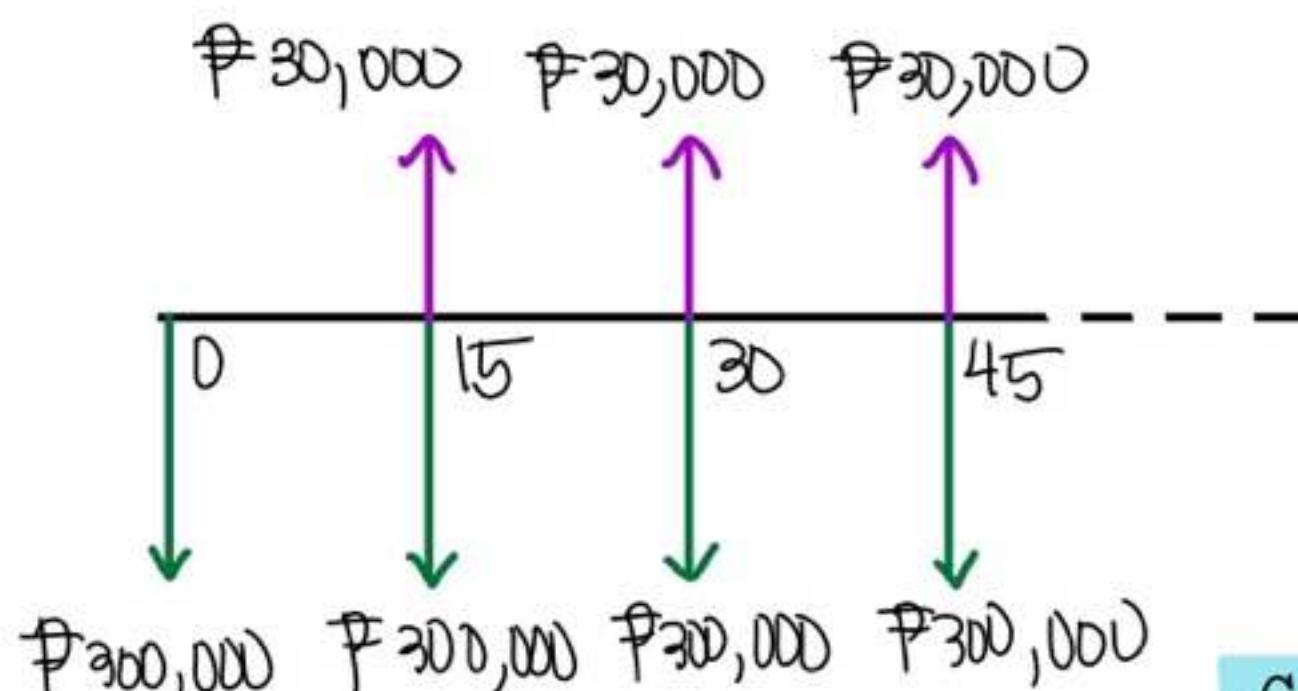
P is the amount invested now at  $i\%$  per period whose interest at the end of every period forever is A while X is the amount invested now at  $i\%$  per period whose interest at the end of every  $k$  period forever is S.

If  $k=1$ , then  $X=P$ .

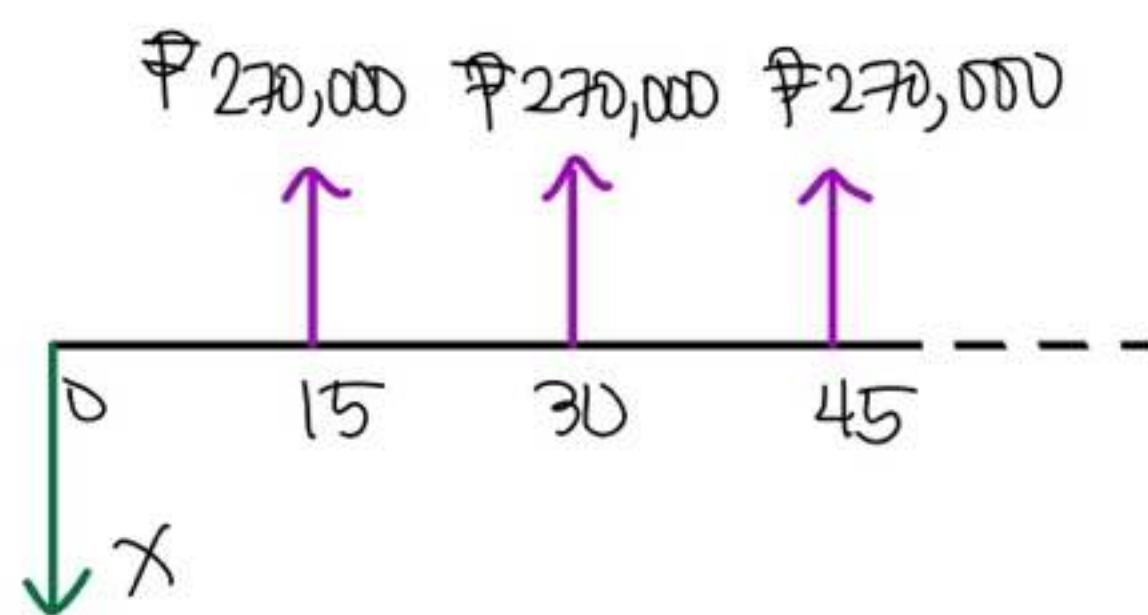
### Sample Problem 21

A new engine was installed by a textile plant at a cost of ₱300,000 and projected to have a useful life of 15 years. At the end of its useful life, it is estimated to have a salvage value of ₱30,000. Determine its capitalized cost if interest is 18% compounded annually.

Solution:



CASH FLOW DIAGRAM



FOR THE ENGINE

$$X = \frac{S}{(1+i)^k - 1} = \frac{\text{P} 270,000}{(1+0.14)^{15} - 1} = \text{P} 24,604$$

$$\begin{aligned}\text{Capitalized Cost} &= \text{First cost} + X \\ &= \text{P} 300,000 + \text{P} 24,604 \\ &= \boxed{\text{P} 324,604}\end{aligned}$$

**Case 3:** Replacement, maintenance and or operation every period

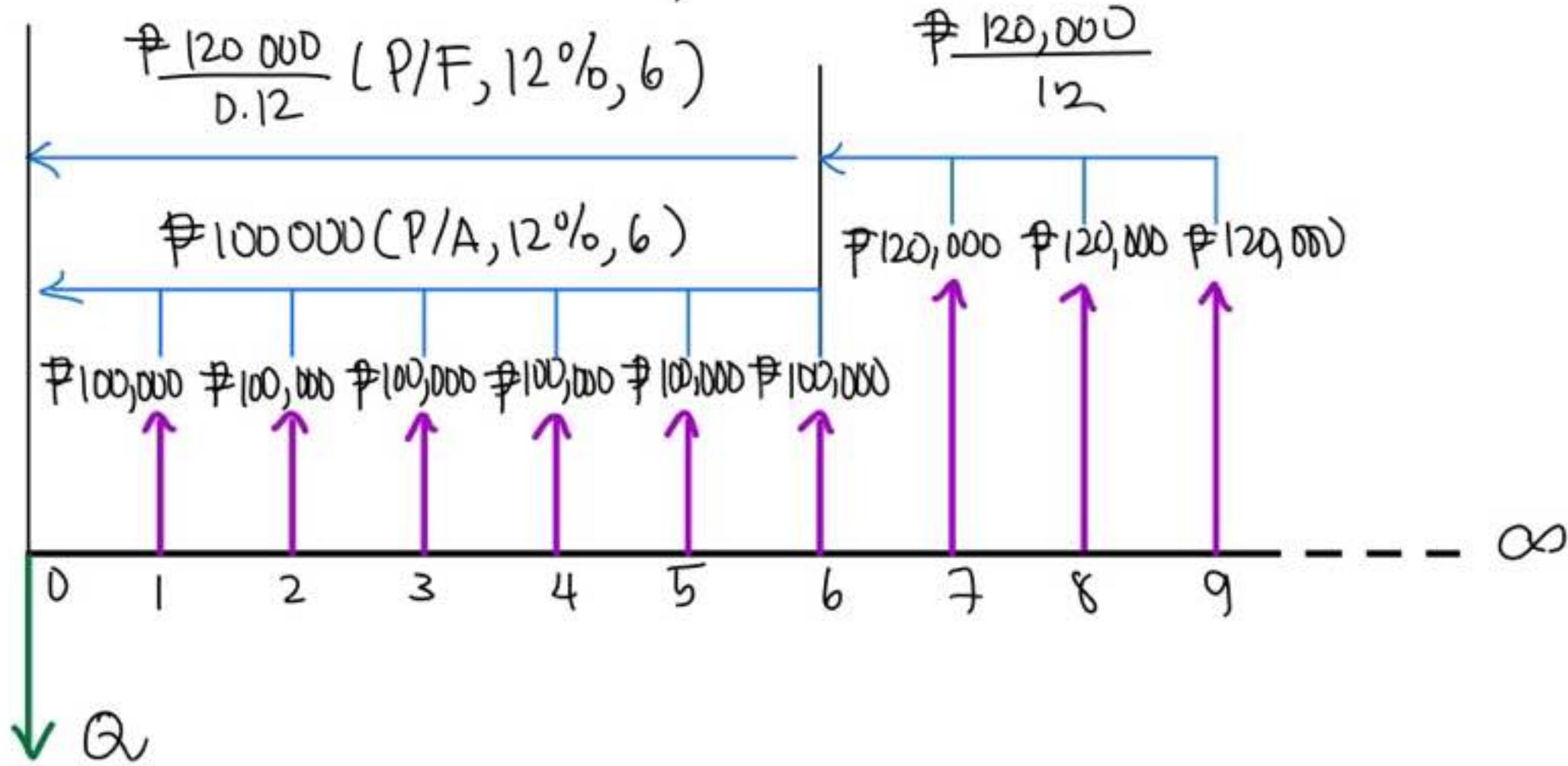
Capitalized Cost = First cost + Present worth of cost of perpetual operation and or maintenance + Present worth of cost of perpetual replacement.

### Sample Problem 22

Determine the capitalized cost of a research laboratory which requires P 5,000,000 for original construction; P 100,000 at the end of every year for the first 6 years and then P 120,000 each year thereafter for operating expenses and P 500,000 every 5 years for replacement of equipment with interest at 12% per annum.

Solution:

Operation:

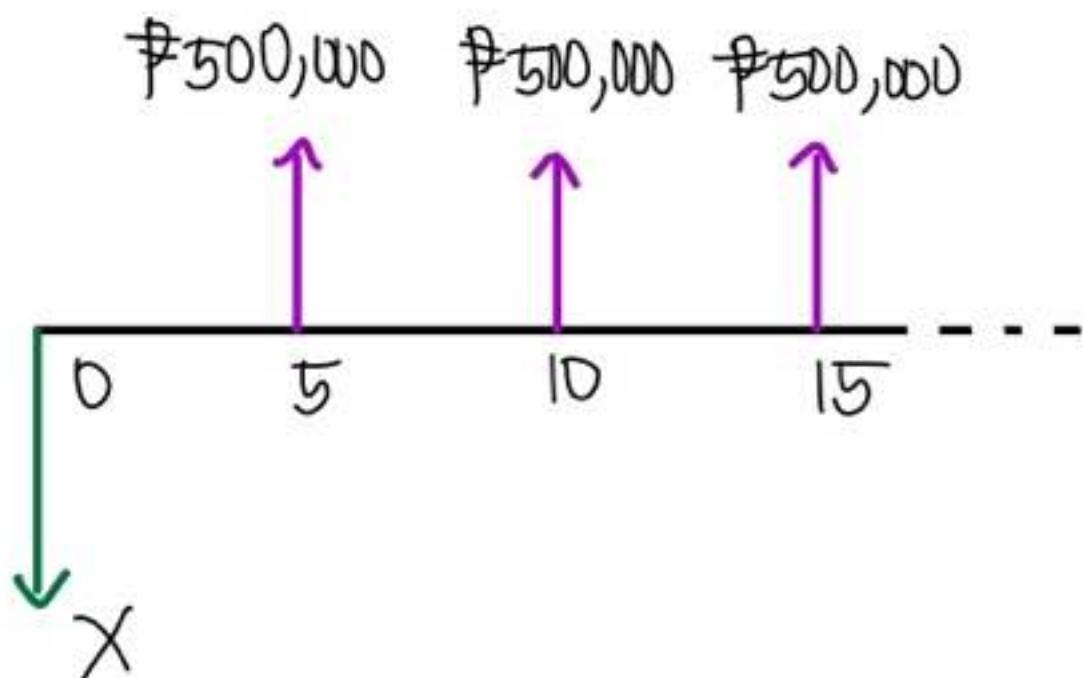


Let  $Q$  be the present worth of cost of perpetual operation

$$Q = \text{P}100,000 \left[ \frac{1 - (1 + 0.12)^{-6}}{0.12} \right] + \text{P}\frac{120,000}{0.12} (1 + 0.12)^{-6}$$

$$Q = \text{P}917,771.85$$

Replacement



Let  $X$  = the present worth of cost of perpetual replacement

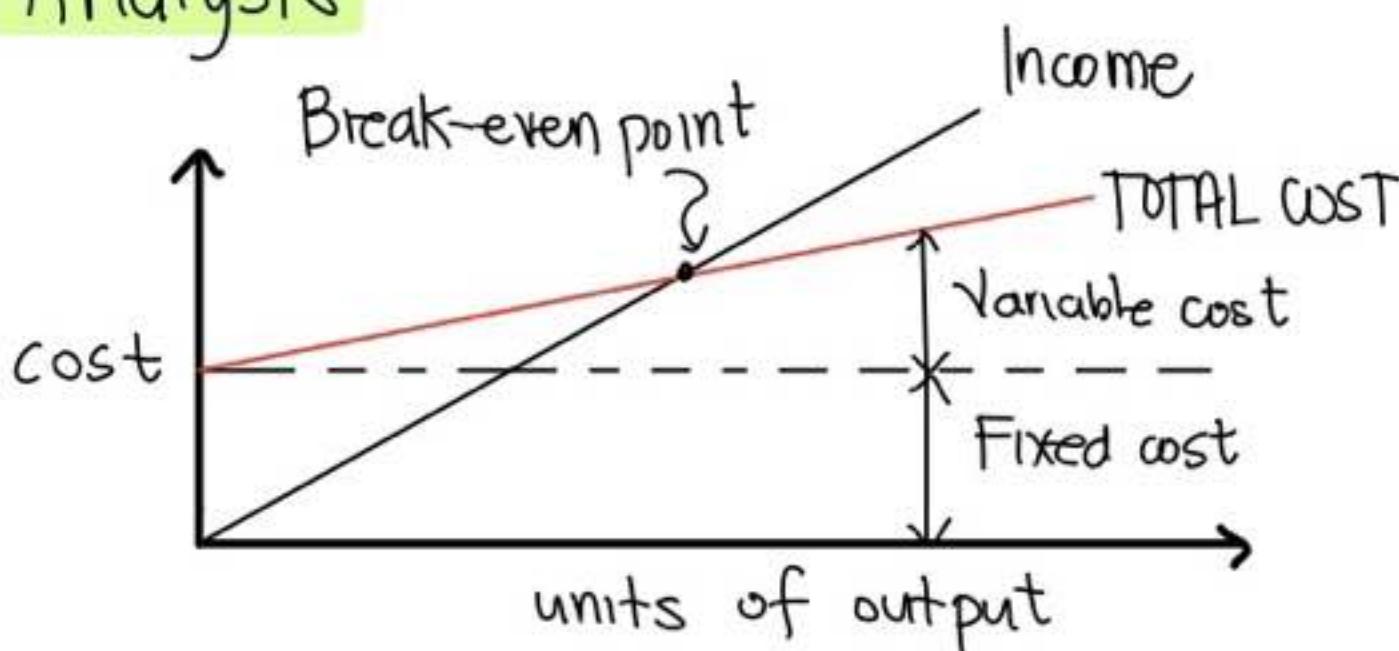
$$X = \frac{S}{(1+i)^k - 1} = \frac{\text{P}500,000}{(1+0.12)^5 - 1} = \text{P}655,873.88$$

Capitalized Cost = First cost +  $Q$  +  $X$

$$= \text{P}5,000,000 + \text{P}917,771.85 + \text{P}655,873.88$$

$$= \boxed{\text{P}6,573,645.73}$$

### Break-Even Analysis



$$\text{Income / sales} = \text{Total cost} = \text{Fixed cost} + \text{Variable cost}$$